Kinematics in HELIOS detector and particle detection

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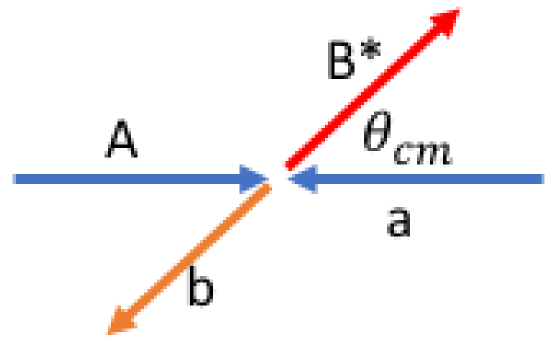
Transfer Reaction

The kinematics of transfer reaction, denote as A(a,b)B, where A is incoming particle with larger mass, a is target nucleus, b and B are scattered particles, in which b is the lighter one.

The four-momenta vectors of particle b and B in the Lab frame are

$$\mathbb{P}_b = \begin{pmatrix} E_b \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \gamma E_{cm} - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta E_{cm} - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E_b \\ p\cos\theta \\ p\sin\theta \end{pmatrix}$$

$$\mathbb{P}_B = \begin{pmatrix} E' \\ p'_z \\ p'_{xy} \end{pmatrix} = \begin{pmatrix} \gamma E'_{cm} + \gamma\beta k \cos\theta_{cm} \\ \gamma\beta E'_{cm} + \gamma k \cos\theta_{cm} \\ -k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E' \\ p'\cos\theta \\ p'\sin\theta \end{pmatrix}$$



where

$$E_{cm} = \sqrt{m_b^2 + k^2} = \frac{1}{2M_c}\left(M_c^2 + m_b^2 - m_B^2\right)$$

$$E'_{cm} = \sqrt{m_B^2 + k^2} = \frac{1}{2M_c}\left(M_c^2 - m_b^2 + m_B^2\right)$$

$$k^2 = \frac{1}{4E_t^2}(M_c^2 - (m_b + m_B)^2)(M_c^2 - (m_b - m_B)^2)$$

$$M_c^2 = (m_a + m_A)^2 + 2m_a T_A$$

$$\beta = \frac{\sqrt{(m_A + T_A)^2 - m_A^2}}{m_a + m_A + T_A}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

In here g (G) is the total energy of particle b (B) in the CM frame, θ_{cm} is the center of mass scattering angle. k is the momentum of particle b or B in the CM frame. E_t is the total energy in the CM frame, or the total invariance mass of the system M_c. T is the total kinetic energy of particle A in the Lab frame. β is the Lorentz boost from the CM frame to the Lab frame, and γ is the Lorentz parameter from β. The momentum of the particle b, in term of lab angle θ, is:

$$p = \frac{\frac{\gamma}{\cos\theta}}{1 + \gamma^2\tan^2\theta}\left(g\beta + \sqrt{k^2 + (k^2 - g^2\beta^2)\gamma^2\tan^2\theta}\right),$$

$$\tan\theta_{cm} = \frac{p\sin\theta}{\beta q - \frac{p}{\gamma}\cos\theta}, \qquad \tan\theta = \frac{1}{\gamma}\frac{k\sin\theta_{cm}}{\beta g - k\cos\theta_{cm}}$$

Special case: the (d,p) reaction at low energy

In a (d,p) reaction, let's make an approximation that $m_A \sim A\, m_u, m_a \sim 2m_u, m_b \sim m_u, m_B \sim (A+1)m_u$, and $T \sim A\,\kappa\, m_u$, where κ is in MeV/m_u, for 10 MeV/u, $\kappa \sim 0.01$. The lab angle for the light particle approximates to

$$\tan\theta \sim \frac{\sin\theta_{cm}}{\frac{\sqrt{2A}}{\sqrt{1+A}} - \cos\theta_{cm}} \rightarrow \frac{\sin\theta_{cm}}{\sqrt{2} - \cos\theta_{cm}}, \qquad when\ \kappa \rightarrow 0$$

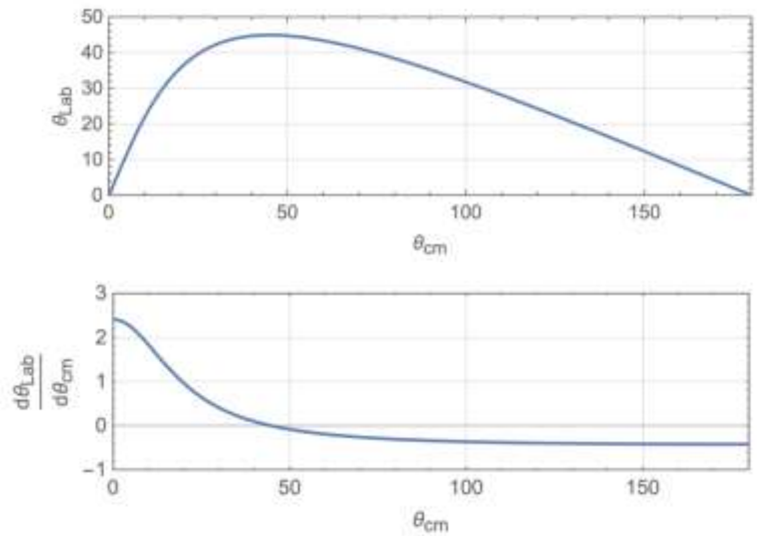


Figure 1 The relation between θ_{cm} and θ_{Lab} for simple approximate for the (d,p) reaction.

The θ_{Lab} (or simply θ) is approximately 2 times than θ_{cm} for $\theta_{cm} < 20°$.

HELIOSmatics

Using this four-momentum vector, we are going to give out the formula that use in HELIOS. The most representation plot is the e – z plot (Figure 2), where the kinetic energy versus position along the HELIOS axis. A typical plot like this:

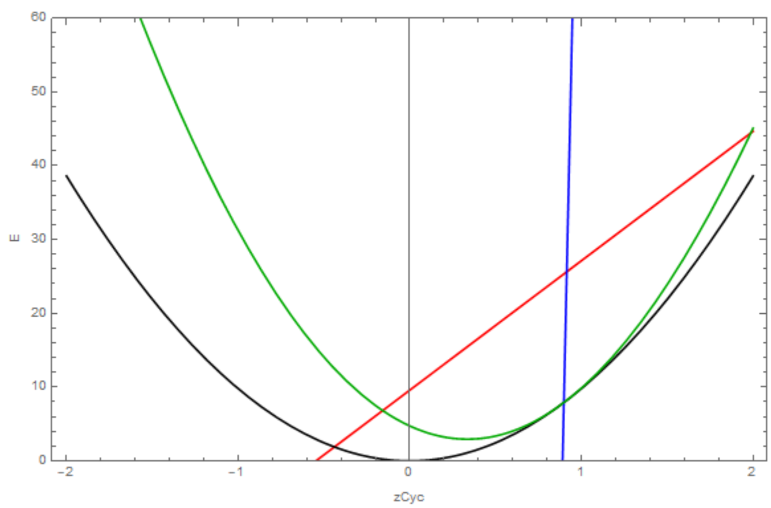


Figure 2 The black curve is the lower or upper bound of the energy or $\theta_{cm} = 0$. The red line the locus for fixed excitation energy (constant E_x line). The blue line is the line for $\theta_{cm} = \pi/2$, and the green curve is constant θ_{cm} or $\pi - \theta_{cm}$.

The basic formula is the cyclotron radius

$$\rho = \frac{P}{cqB},$$

where P is momentum in MeV/c that perpendicular to the magnetic field B (in Tesla), q is the charge state, $c = 299.792458$. the unit of ρ is meter. Under the kinematics of transfer reaction

$$\rho = \frac{p_{xy}}{cqB} = \frac{k \sin\theta_{cm}}{cqB} \; [meter]$$

The time for a cycle is

$$t = \frac{2\pi\rho}{v_\perp} = \frac{2\pi}{cqB}\frac{k\sin\theta_{cm}}{v_\perp} \; [sec]$$

The time for a cycle is almost fixed. Thus, the length for a cycle is

$$\begin{aligned} z &= v_\parallel t = 2\pi\rho\frac{v_\parallel}{v_\perp} \\ &= \frac{2\pi}{cqB}\frac{v_\parallel}{v_\perp}k\sin\theta_{cm}, \qquad \frac{v_\parallel}{v_\perp} = \frac{1}{\tan\theta} \\ &= 2\pi\frac{\rho}{\tan\theta} \\ &= \frac{2\pi}{cqB}p_z \end{aligned}$$

We have

$$\alpha\, z = p_z = \gamma\beta E_{cm} - \gamma k\cos\theta_{cm}, \qquad \alpha = \frac{cqB}{2\pi}$$

With the energy equation, we have 2 coupled equations (*master coupled equations*):

$$\begin{aligned} \alpha\, z &= \gamma\beta E_{cm} - \gamma k\cos\theta_{cm} \\ E_b &= \gamma E_{cm} - \gamma\beta k\cos\theta_{cm} \end{aligned}$$

By eliminating difference variables, we can get all difference curves or lines.

Cyclotron period

$$\begin{aligned} t_c &= \frac{2\pi\rho}{v_\perp} \\ &= \frac{2\pi}{cqB}\frac{k\sin\theta_{cm}}{v_\perp} \\ t_c &= \frac{2\pi}{cB}\frac{m}{q}\gamma_L \end{aligned}$$

The $k\sin(\theta_{cm})$ is same as the perpendicular component of the momentum in the Lab frame, i.e. p_{xy}.

$$p_{xy} = p_\perp = m v_\perp \gamma_L$$

Where γ_L is the Lab frame Lorentz gamma.

The constant E_x line

First, by eliminating $\cos\theta_{cm}$ in the master coupled equations, we get the red line in Figure 2, which only depends on excitation energy

$$e = \frac{E_{cm}}{\gamma} - m_b + \alpha\beta z$$
$$e = \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma M_c} - m_b + \frac{cqB}{2\pi}\beta z$$

The intercept of the red line is

$$e_0 = \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma M_c} - m_b$$

The only non-constant is m_B, which can be excited. Let examine the term, for $E_x \ll m_B$

$$\frac{m_B^2}{2\gamma M_c} \to \frac{(m_B + E_x)^2}{2\gamma M_c} \approx \frac{m_B^2}{2\gamma M_c}\left(1 + \frac{2E_x}{m_B}\right) = \frac{m_B^2}{2\gamma M_c} + \frac{m_B}{\gamma M_c}E_x$$

At small incident energy, $M_c = m_b + m_B + T_{cm} \approx m_B, \gamma \approx 1$,

$$e_0 \approx \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma M_c} - m_b - E_x$$

Second, we can also eliminate e, so that,

$$\cos\theta_{cm} = \frac{\beta E_{cm}}{k} - \frac{\alpha}{\gamma k} z$$

This is the center-of-mass-angle to z-position relationship. The dependency of the excitation energy is inside the term E_{cm}.

The constant θ_{cm} line

Next, we eliminate m_B from the master coupled equations. Notice that m_B is implicitly contained inside $k = k(m_B)$, we have a complicated curve

$$e = -m_b + \frac{-\sin^2(\theta_{cm})\,\alpha\beta\gamma^2 z + \cos\theta_{cm}\sqrt{\alpha^2 z^2 + m_b^2(1 - \sin^2(\theta_{cm})\,\gamma^2)}}{1 - \sin^2(\theta_{cm})\,\gamma^2}$$

This is a general contour for a given θ_{cm}. When $\theta_{cm} = 0$, it reduces to

$$e = -m_b + \sqrt{\alpha^2 z^2 + m_b^2}$$

This is the black curve in Figure 2. When $\theta_{cm} = \frac{\pi}{2}$,

$$e = -m_b + \frac{\alpha}{\beta} z$$

This is the blue line in Figure 2.

The Bore radius line

Since the detector may have maximum radius R, and $2\rho \leq R$. Thus,

$$\rho = \frac{k \sin\theta_{cm}}{cqB} \leq \frac{R}{2} \Rightarrow k \sin\theta_0 = R\frac{cqB}{2} = R\alpha\pi$$

Put in the 2 coupled equations:

$$\alpha\, z = \gamma\beta E_{cm} - \gamma\sqrt{k^2 - (R\alpha\pi)^2}$$
$$e + m_b = \gamma E_{cm} - \gamma\beta\sqrt{k^2 - (R\alpha\pi)^2}$$

Expand in recoil mass m_B,

$$k^2 = \frac{1}{4M_c^2}(M_c^2 - (m_b + m_B)^2)(M_c^2 - (m_b - m_B)^2)$$

$$\alpha\, z = \gamma\beta \frac{1}{2M_c}\left(M_c^2 + m_b^2 - m_B^2\right) - \gamma\sqrt{\frac{1}{4M_c^2}(M_c^2 - (m_b + m_B)^2)(M_c^2 - (m_b - m_B)^2) - (R\alpha\pi)^2}$$

$$e + m_b = \gamma \frac{1}{2E_t}\left(M_c^2 + m_b^2 - m_B^2\right) - \gamma\beta\sqrt{\frac{1}{4M_c^2}(M_c^2 - (m_b + m_B)^2)(M_c^2 - (m_b - m_B)^2) - (R\alpha\pi)^2}$$

Eliminate m_B

$$2m_b e + e^2 = \alpha^2(\pi^2 R^2 + z^2)$$

$$e = \sqrt{\alpha^2(\pi^2 R^2 + z^2) + m_b^2} - m_b$$

Compare with the constant $\theta_{cm} = 0$ line

$$e = \sqrt{\alpha^2 z^2 + m_b^2} - m_b^2$$

Maximum excitation energy

We can see that, when the excitation energy of particle B is higher, the red line shifts lower, there is an upper limit for the red line to be shifted, which is when the red line touches the black curve.

$$(e_{max}, z_{max}) = \left(m_b\gamma - m_b, \frac{\gamma\beta}{\alpha} m_b\right)$$

Solve for the maximum m_B, in the CM frame,

$$m_B(\max) = m_b + E_x(\max) = M_c - m_b = \sqrt{(m_a + m_A)^2 + 2m_a T_A} - m_b$$

At non-relativistic limit,

$$m_B(\max) = \sqrt{(m_a + m_A)^2 + 2m_a T_A} - m_b \to m_a + m_A + \frac{m_a T_A}{m_a + m_A} - m_b = Q + m_B + \frac{m_a T_A}{m_a + m_A}$$

$$E_x(\max) = Q + \frac{m_a}{m_a + m_A} T_A = Q + T_{cm}$$

where T_{cm} is the CM frame kinematics energy. In fact, it is easily to prove that

$$T_{cm} = \frac{m_a}{m_a + m_A} T_A$$

Minimum Incident energy

The minimum incident energy requires that $k \geq 0$, thus

$$k^2 = \frac{1}{4M_c^2}(M_c^2 - (m_b + m_B)^2)(M_c^2 - (m_b - m_B)^2) \geq 0$$

The most minimum factor is

$$\begin{aligned} & M_c^2 - (m_b + m_B)^2 \geq 0 \\ \Rightarrow \quad & M_c \geq m_b + m_B \\ \Rightarrow \quad & (m_a + m_A)^2 + 2m_a T_{min} = (m_b + m_B)^2 \end{aligned}$$

$$\begin{aligned} T_{min} &= \frac{(m_b + m_B)^2 - (m_a + m_A)^2}{2m_a} \\ &\cong -Q\left(1 + \frac{m_A}{m_a}\right) \neq Q \end{aligned}$$

We can divide the minimum incident energy by the mass number of particle A, replace $m_A \approx A\, m_u, m_a \approx a\, m_u$, we have

$$\frac{T_{min}}{A} \cong -Q\left(\frac{a + A}{aA}\right) = -\frac{Q}{\min(a, A)}$$

Tilted Reaction

When the incident particle makes some incident angle θ_A, the four-momentum of particle b will be tilted by angle θ_A,

$$\mathbb{P}_b = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_A & -\sin\theta_A \\ 0 & \sin\theta_A & \cos\theta_A \end{pmatrix} \begin{pmatrix} \gamma g - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta g - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix}$$

Since the z-position is

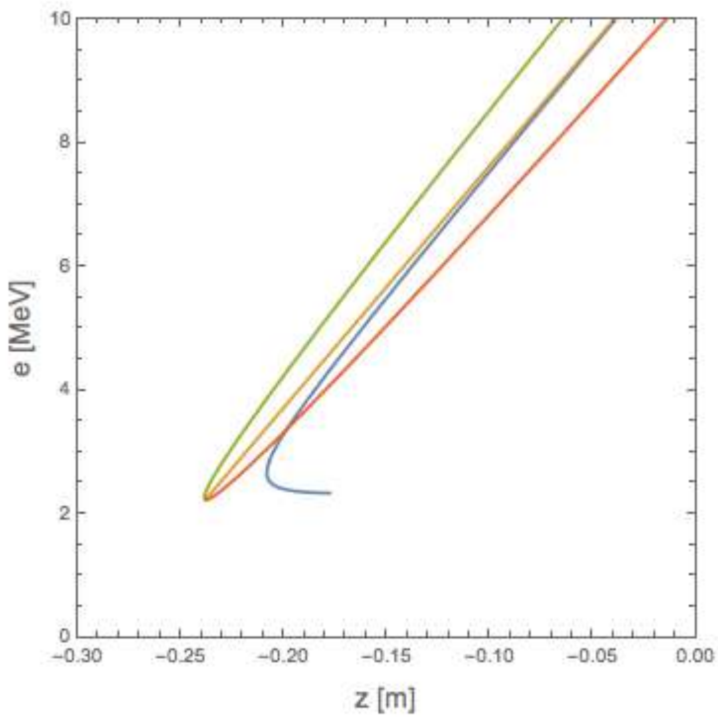
$$\alpha\, z = p_z = (\gamma\beta E_{cm} - \gamma k \cos\theta_{cm}) \cos\theta_A + k \sin(\theta_{cm}) \sin\theta_A$$

With the energy

$$e + m = E_{cm} - \gamma\beta k \cos\theta_{cm}$$

Eliminate θ_{cm}, we get

$$\alpha\beta z = \left(e + m - \frac{E_{cm}}{\gamma}\right)\cos\theta_A + \frac{1}{\gamma}\sqrt{(\gamma\beta k)^2 - (E_{cm} - e - m)^2}\sin\theta_A$$



In the above plot, the orange line is the normal constant E_x line, the green curve is $\theta_A = 50$ mrad, and the red curve is $\theta_A = -50$ mrad. 50 mard ~ 2.9 deg. And the blue curve is with finite detector correction, such that $a = 0.01$ meter. The reaction is ^{208}Pb(d,p) at 10 MeV/u, magnetic field is 2.85 T.

In this calculation, we can see the finite emittance of the beam could contribute a lot to the energy resolution.

Classical Limit

When the beam energy is small so that the relativistic effect is small, $\beta c \to V_{cm}$, the cyclotron period becomes

$$t_c = \frac{2\pi}{cB}\frac{m}{q}\gamma_L \to \frac{2\pi}{cB}\frac{m}{q}$$

The constant E_x line is

$$E_b = \frac{E_{cm}}{\gamma} + \frac{cqB}{2m}\beta z \to E_b = \frac{E_{cm}}{\gamma} + \frac{mV_{cm}}{t_c}z$$

The CM frame energy $E_{cm} \to mc^2 + \frac{1}{2}mu^2$. Divided by the Lorentz γ, it becomes

$$\frac{E_{cm}}{\gamma} \to mc^2 + \frac{1}{2}mu^2 - \frac{1}{2}mV_{cm}^2$$

Also, $E_b \to mc^2 + e$

$$e \cong \frac{1}{2}mu^2 - \frac{1}{2}mV_{cm}^2 + \frac{mV_{cm}}{t_c}z$$

Alpha Source

When alpha source is put at the axis, the 4-momentum is

$$\mathbb{P} = (E, p_z, p_{xy}), \qquad E = m_\alpha + T, \qquad p_z = p\cos\theta\,, \qquad p = \sqrt{2m_\alpha T + T^2}$$

Under a magnetic field, the bending radius is

$$\rho = \frac{P}{cqB}$$

where P is momentum in MeV/c that perpendicular to the magnetic field B (in T), q is the charge state, $c = 299.792458$. The unit of ρ is meter

$$\rho = \frac{p\sin\theta}{cqB},$$

The time for a cycle is

$$t = \frac{2\pi\rho}{v_\perp} = \frac{2\pi}{cqB}\frac{p\sin\theta}{v_\perp}$$

Thus, the length for a cycle is

$$z_0 = v_\parallel t = 2\pi\rho\frac{v_\parallel}{v_\perp} = \frac{2\pi}{cqB}\frac{v_\parallel}{v_\perp}p\sin\theta = \frac{2\pi}{cqB}p\cos\theta\,, \qquad \frac{v_\parallel}{v_\perp} = \frac{1}{\tan\theta}$$

$$z_0 = \frac{2\pi}{cqB}p\cos\theta$$

The locus is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} \sin\left(\tan(\theta)\frac{z}{\rho} - \phi\right) + \sin\phi \\ \sigma\left(\cos\left(\tan(\theta)\frac{z}{\rho} - \phi\right) - \cos\phi\right) \end{pmatrix}$$

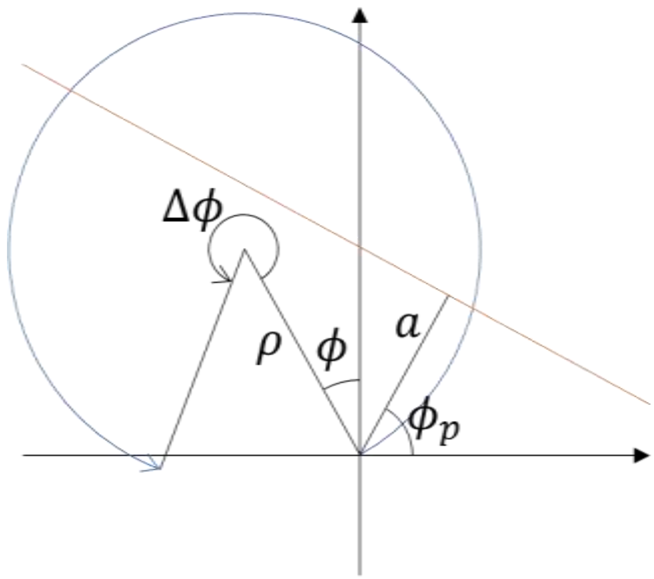
Where $\sigma = +1$ when B-field is parallel with the z-axis for positive charged particle. $\sigma = -1$ when the B-field is anti-parallel.

The radius is

$$r = \sqrt{x^2 + y^2} = \sqrt{2}\rho\sqrt{1 - \cos\left(\tan(\theta)\frac{z}{\rho}\right)}$$

Finite axial detector

A finite axial detector is a polygonal prism that surrounded and centered the HELIOS axis and larger than the beam size. The blue circle is the XY projection of the particle trajectory. The orange line is one of the detector plans.



For am axial detector, the normal of a plane is

$$\hat{n} = \left(\cos\phi_p\,, \sin\phi_p\,, 0\right)$$

The equation for the detector plane is

$$x\cos\phi_p + y\sin\phi_p = a$$

Where a is the shortest distance from the plane to z-axis.

The equation of the locus of the positive charged particle, when the B-field is direct to the z-axis, is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} \sin\left(\tan(\theta)\frac{z}{\rho} - \phi\right) + \sin\phi \\ \sigma\left(\cos\left(\tan(\theta)\frac{z}{\rho} - \phi\right) - \cos\phi\right) \end{pmatrix}$$

Where (θ, ϕ) is the scattering angle of positively charged particle b, σ is the +1 for B-field along z-axis, -1 for B-field against z-axis.

Define $\Delta\phi = \phi - \phi_p$, the hit points are

$$z_{hit} = \frac{\rho}{\tan(\theta)}\left(\sigma\Delta\phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} - \sigma\sin(\Delta\phi)\right)\right), n = 0,1,2 \ldots.$$

For real solution,

$$-1 < \frac{a}{\rho} - \sigma\sin(\Delta\phi) < 1$$

Notice that the length for a cycle is

$$z_0 = \frac{2\pi\,\rho}{\tan(\theta)}$$

$$z_{hit} = \frac{z_0}{2\pi}\left(\sigma\Delta\phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} - \sigma\sin(\Delta\phi)\right)\right)$$

Since we want to know which point is hit from outside, i.e. the direction of the particle is toward the axis, not outward from the axis.

The direction vector for the particle is

$$\frac{d}{dz}\begin{pmatrix} x \\ y \end{pmatrix} = \rho\tan(\theta)\begin{pmatrix} \cos\left(\tan(\theta)\frac{z}{\rho} - \sigma\phi\right) \\ \sigma\sin\left(\tan(\theta)\frac{z}{\rho} - \sigma\phi\right) \end{pmatrix}$$

The dot product with the plane normal

$$\cos\theta' = \rho\tan(\theta)\cos\left(\tan(\theta)\frac{z}{\rho} - \sigma\Delta\phi\right) < 0$$

$$\Rightarrow \cos\left(\tan(\theta)\frac{z}{\rho} - \sigma\Delta\phi\right) < 0$$

In fact, using geometrical argument, for $n = odd$ number, the hit point is always inward. Substitute z_{hit}

$$\begin{aligned} &\cos\left(n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} - \sigma\sin(\Delta\phi)\right)\right) \\ &= (-1)^n \cos\left(\sin^{-1}\left(\frac{a}{\rho} - \sigma\sin(\Delta\phi)\right)\right) \\ &= (-1)^n \sqrt{1 - \left(\frac{a}{\rho} - \sigma\sin(\Delta\phi)\right)^2} < 0 \end{aligned}$$

This proves the geometrical argument that $n = odd$.

A special case for $\phi = 0, \phi_p = \pi, n = 1$

$$\begin{aligned} z_{hit} &= \frac{\rho}{\tan(\theta)}\left(2\pi - \sin^{-1}\left(\frac{a}{\rho}\right)\right) \\ &= z_0\left(1 - \frac{1}{2\pi}\sin^{-1}\left(\frac{a}{\rho}\right)\right) \end{aligned}$$

The rotated angle is smaller then 2π. When $\rho \gg a$

$$z_{hit} \approx z_0\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right)$$

With the transfer reaction, when e is large, $\rho \gg a$, and the $\sin^{-1}()$ becomes small. Express ρ and z_0

$$\rho = \frac{k\sin\theta_{cm}}{cqB}, \qquad \alpha\beta z_0 = E_b - \frac{E_{cm}}{\gamma}$$

Since

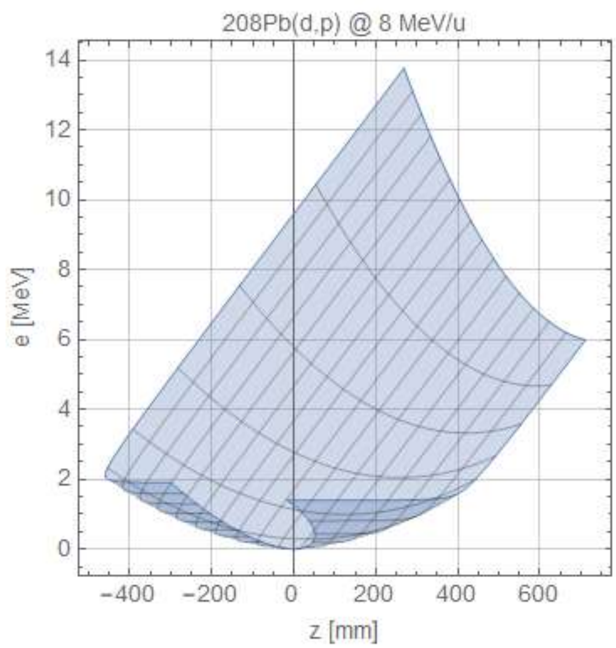
$$E_b = \gamma E_{cm} - \gamma\beta k\cos\theta_{cm}$$
$$\Rightarrow \cos\theta_{cm} = \frac{1}{\gamma\beta k}(\gamma E_{cm} - E_b)$$

Thus,

$$k\sin\theta_{cm} = \frac{\sqrt{\gamma^2\beta^2k^2 - (\gamma E_{cm} - E_b)^2}}{\gamma\beta}$$

Than we have,

$$\alpha\beta z = \left(E_b - \frac{E_{cm}}{\gamma}\right)\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{\gamma^2\beta^2k^2 - (\gamma E_{cm} - E_b)^2}}\right)$$



Off-axis Effect

When the helix of particle b is off axis.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho\begin{pmatrix} \sin\left(\tan(\theta)\frac{z}{\rho} + \phi\right) - \sin\phi \\ \cos\phi - \cos\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \end{pmatrix} + \rho_0\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

The solution for z_{hit} becomes

$$z_{hit} = \frac{\rho}{\tan(\theta)}\left(\phi_p - \phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a - \rho_0\cos(\phi_0 - \phi_p)}{\rho} + \sin(\phi - \phi_p)\right)\right), n = 0,1,2 \ldots.$$

For $n = 1, \phi_p = \pi, \phi = 0$

$$z_{hit} = z_0\left(1 - \frac{1}{2\pi}\sin^{-1}\left(\frac{a + \rho_0\cos(\phi_0)}{\rho}\right)\right), \qquad \left|\frac{a + \rho_0\cos(\phi_0)}{\rho}\right| < 1$$

We can see, the effect is same as change of the effective $a_{eff} = a - \rho_0\cos(\phi_0 - \phi_p)$. Also the beam size must be smaller than the detector distance $\rho > a > \rho_0\cos(\phi_0 - \phi_p)$.

Radial Detector (no conclusion yet)

A radial detector is a plane detector located at fixed z-pos (z_R) and perpendicular to HELIOS axis. The hit position with an off-axis helix is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho\begin{pmatrix} \sin\left(\tan(\theta)\frac{z_R}{\rho} + \phi\right) - \sin\phi \\ \cos\phi - \cos\left(\tan(\theta)\frac{z_R}{\rho} + \phi\right) \end{pmatrix} + \rho_0\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

For $\rho_0 = 0$, the radial position is

$$r = \sqrt{x^2 + y^2} = \rho\sqrt{2 - 2\cos\left(\tan(\theta)\frac{z_R}{\rho}\right)}$$

$$\rho = \frac{p_{xy}}{cZB} = \frac{k\sin\theta_{cm}}{cZB}, \qquad \tan\theta = \frac{p_{xy}}{p_z}, \qquad \frac{\tan(\theta)\, z_R}{\rho} = \frac{cZB}{p_z} z_R$$

Since the z-pos is fixed, the TOF from target to the detector is

$$t = \frac{z_R}{v_z} = \frac{z_R}{c}\frac{E}{p_z}$$

Eliminate the $k\cos\theta_{cm}$ in p_z, we have

$$e + m = \frac{E_{cm}}{\gamma}\frac{ct}{\beta z_R - ct}$$

Knockout reaction

The reaction is notated as A(a, 12)B, where A = B + b, in which b is the bounded nucleus, and 1 and 2 are free scattering particles. When particle b knocked out, it becomes particle 2. The energy and momentum conservation are

$$\mathbb{P}_A + \mathbb{P}_a = \mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_B$$

In which the mass of particle B is

$$m_B + m_2 = m_A + S$$

The reaction Q-value is

$$Q = m_A + m_a - m_1 - m_2 - m_B = -S$$

The recoil of the particle B assumed the form

$$\mathbb{P}_B = (m_B, -\overrightarrow{k_b}) = \left(\sqrt{(m_A - m_2 + S)^2 + |\overrightarrow{k_b}|^2}, -\overrightarrow{k_b}\right)$$

Where S is separation energy, $\vec{k}$ is the recoiled momentum, which is same but opposite direction with the bounded nucleus b, as particle A is stationary.

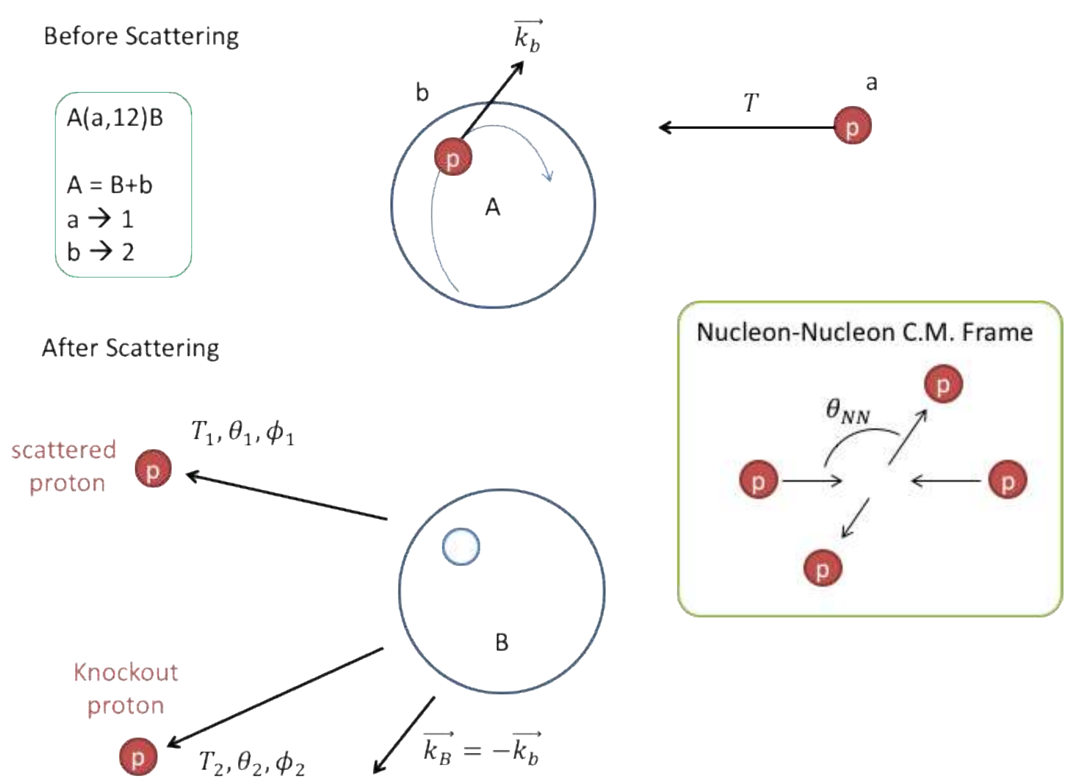
Assume a is the incident particle and A is the target, We can from a quai-particle b by

$$\mathbb{P}_b = \mathbb{P}_A - \mathbb{P}_B$$

$$\mathbb{P}_b = (m_b, \vec{k}_b) = \left(m_A - \sqrt{(m_A - m_a + S)^2 + |\overrightarrow{k_b}|^2}, \overrightarrow{k_b}\right)$$

Thus, the rest is like that of transfer reaction, except the target is also moving.

Because the "target" b and incident particle a are both moving, this forms the plane of incident channel. And the exit channel, particles 1 and 2 cannot be on the same plane. The following illustration is normal kinematics.



Once the quasi-particle is constructed, the reaction is reduced to

$$\mathbb{P}_a + \mathbb{P}_b = \mathbb{P}_1 + \mathbb{P}_2$$

Thus, the next step of calculation is identical to transfer reaction. The reaction constants are

$$\vec{\beta} = \frac{\overrightarrow{k_a} + \overrightarrow{k_b}}{E_a + E_b}, \qquad \gamma = \frac{1}{\sqrt{1 - |\beta|^2}}$$

$$E_t = \sqrt{(E_a + E_b)^2 + \left|\overrightarrow{k_a} + \overrightarrow{k_b}\right|^2}$$

$$k^2 = \frac{1}{4E_t^2}(E_t^2 - (m_1 + m_2)^2)(E_t^2 - (m_1 - m_2)^2)$$

in above, $\vec{\beta}$ is the Lorentz boost to NN-CM frame. E_t is the total energy in NN-CM frame, or the intrinsic total energy of NN-system. k is the magnitude of momentum of the scattered particles 1 and 2 in NN-CM frame.

Since the Lorentz boost of from the Lab frame to the NN-CM (nucleon-nucleon center of mass) frame is not on the z-axis, the formula for particles 1 and 2 is complicated. In the NN-CM frame, the four-vector for particle 1 is

$$\mathbb{P}_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \sqrt{m_1^2 + k^2} \\ k\cos\theta \\ k\sin\theta \end{pmatrix}$$

Where θ is not the CM frame scattering angle, because the particle a could has some finite polar angle.

Inverse Kinematics

In inverse kinematics, the momentum $\overrightarrow{k_a} = 0$, that simplify the calculation that, the reaction is a tilted transfer reaction, i.e. the reaction axis is not on the z-axis.

Reconstruct scattered four-momentum

In the knockout experiment, we needed to reconstruct the four momenta. Under HELIOS, to problem is converting z_{hit} to θ_i, the lab angle.

Inverse Problem

We show that the solution from CM frame to Lab frame, or from theory to experiment. Basically, the HELIOS is a problem of finding the mapping

$$\begin{pmatrix} E_x \\ \theta_{cm} \end{pmatrix} \leftrightarrow \begin{pmatrix} e \\ z \end{pmatrix}$$

In term of E_x and $\cos\theta_{cm}$

We can express (z, e) in term of (E_x, θ_{cm}) as

$$\binom{e}{z} = \frac{\gamma}{2M_c}\begin{pmatrix} M_c^2 + m_b^2 - (m_B + E_x)^2 - \beta\cos\theta_{cm}\sqrt{(M_c^2 - (m + m_B + E_x)^2)(M_c^2 - (m - M - E_x)^2)} \\ \beta(M_c^2 + m_b^2 - (m_B + E_x)^2) - \cos\theta_{cm}\sqrt{(M_c^2 - (m + M + E_x)^2)(M_c^2 - (m - M - E_x)^2)} \end{pmatrix}$$

The inverse

$$\begin{pmatrix} E_x \\ \cos\theta_{cm} \end{pmatrix} = \begin{pmatrix} -m_B + \sqrt{M_c^2 + m_b^2 - 2\gamma M_c(E_b - \alpha\beta z)} \\ \dfrac{\gamma(E_b\beta - \alpha\, z)}{\sqrt{\gamma^2(E_b - \alpha\beta z)^2 - m_b^2}} \end{pmatrix}$$

Get k and θ_{cm} from e and z

From experiment, we get the energy ($y = E_b = e + m$) and position (z), then we can reconstruct the reaction constant k and θ_{cm}.

$$k^2 = \gamma^2(y - \beta\alpha z)^2 - m^2$$

$$\cos\theta_{cm} = \frac{(\alpha z - \beta y)}{\gamma k} = \frac{\gamma\sqrt{m^2 + k^2} - y}{\gamma\beta k}$$

From k^2, the total mass of the particle B is

$$m_B^2 = m_b^2 + M_c^2 - 2M_c\sqrt{k^2 + m_b^2}$$

Where M_c is the total mass of the system.

Finite detector

The coupled solution is

$$y = e + m = \gamma\sqrt{m^2 + k^2} - \gamma\beta k\cos\theta_{cm}$$

$$\rho = \frac{k\sin\theta_{cm}}{cqB} = \frac{k\sin\theta_{cm}}{2\pi\alpha}$$

$$\alpha\beta\gamma z = \left(\gamma y - \sqrt{m^2 + k^2}\right)\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right)$$

Solve $\cos\theta$ from 1st equation, and sub into 2nd equation,

$$\alpha\beta\gamma z = \left(\gamma y - \sqrt{m^2 + k^2}\right)\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2y\gamma\sqrt{m^2 + k^2} - y^2 - m^2\gamma^2 - k^2}}\right)$$

Use

$$k \to m\tan(x)\,, \qquad 0 < x < \frac{\pi}{2}$$

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\tan^2(x)}}\right)$$

Under the square root,

$$\begin{aligned}
&2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\sec^2(x) + m^2 \\
&= -y^2\gamma^2 + 2y\gamma m\sec(x) - m^2\sec^2(x) + y^2\gamma^2 - y^2 - m^2\gamma^2 + m^2 \\
&= -(y\gamma - m\sec(x))^2 + (y^2 - m^2)\gamma^2\beta^2
\end{aligned}$$

Than

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{(y^2 - m^2)\gamma^2\beta^2 - (y\gamma - m\sec(x))^2}}\right)$$

Replace

$$\begin{aligned}
&y\gamma - m\sec(x) \to K \\
&(y^2 - m^2)\gamma^2\beta^2 \to H^2 > 0 \\
&\alpha\beta\gamma z \to Z \\
&\beta\gamma\alpha a \to G > 0
\end{aligned}$$

$$Z = K\left(1 - \frac{G}{\sqrt{H^2 - K^2}}\right)$$

Next, replace

$$K \to H\sin\phi\,, \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

$$Z = H\sin\phi\left(1 - \frac{G}{H\cos\phi}\right)$$

or

$$Z = H\sin\phi - G\tan\phi$$

The momentum square is

$$k^2 = (\gamma E_{cm} - H\sin\phi)^2 - m^2$$

When $a \to 0, G \to 0$

$$\begin{aligned}
&Z = H\sin\phi = K = \gamma y - m\sec(x) \\
&\to \alpha\beta\gamma z = \gamma y - \sqrt{m^2 + k^2}
\end{aligned}$$

$$\rightarrow y = \frac{1}{\gamma}\sqrt{m^2 + k^2} + \alpha\beta z$$

Or

$$k^2 = \gamma^2 (y - \alpha\beta z)^2 - m^2$$

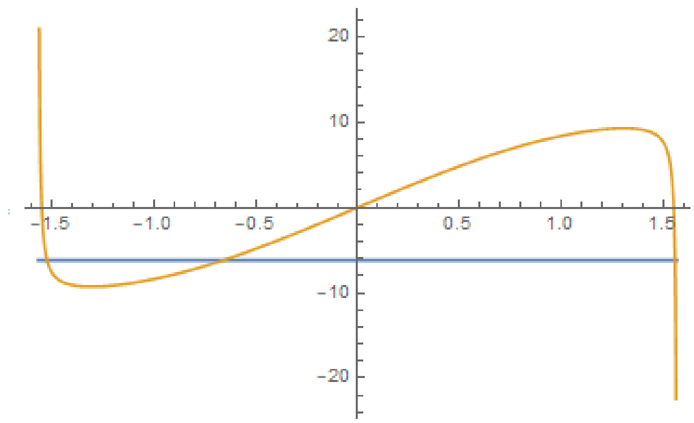
Return to the infinite detector solution.

Since $H, G > 0$, and $G < H$, as the term $\frac{G}{\sqrt{H^2 - K^2}} = \frac{a}{2\pi\rho} < 1$

The function

$$f(\phi) = H \sin\phi - G \tan\,\phi$$

Looks like this

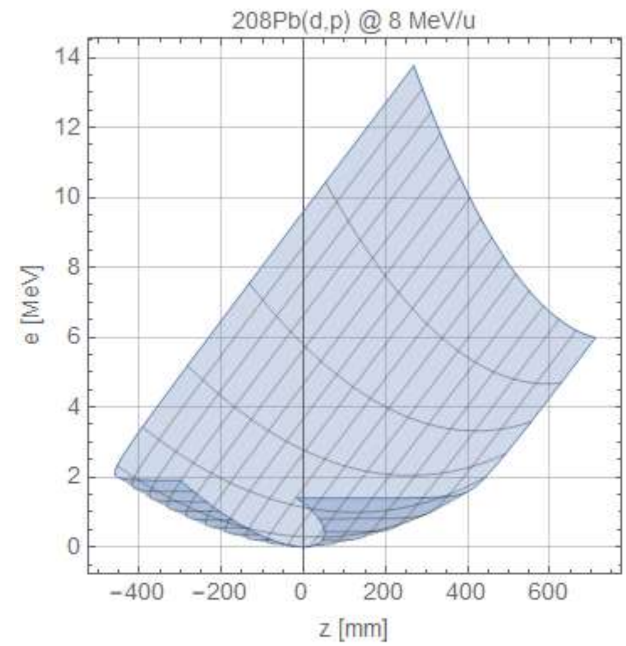


Where the orange line is the $f(\phi)$ and the blue line is $f(\phi) = Z$. We can see, there are multi-solution for ϕ. Normally, when $\theta_{cm} \gg 0$, the derivative is

$$f'(\phi) = H \cos\phi - G \sec^2\phi > 0$$

From experience (need proof), $f'(\phi) > 0$ is the correct solution for most of the case.

Only when the θ_{cm} is too small, so that the e-z line bended so much. In the following plot, the $E_x \in (0, 15)$ MeV, $\theta_{cm} \in (0°, 60°)$. There is region where double solution exists, in that case, at the boundary that manifold is folded, the proper solution takes $f'(\phi) < 0, \phi\, Z > 0$.



It seems (Need proof) that the fold happens when

$$\gamma^2\beta^2k^2 - (\gamma E_{cm} - y)^2 = 0$$

To numerically find the solution, a newton's method is adequacy.

$$\phi_{i+1} = \phi_i - \frac{f(\phi_i)}{f'(\phi_i)}$$

Appendix

Lorentz Transform with boost $\vec{\beta}$

$$\mathbb{P} = \begin{pmatrix} E \\ \vec{k} \end{pmatrix} \rightarrow \mathbb{P}' = \begin{pmatrix} \gamma E + \gamma\vec{\beta}\cdot\vec{k} \\ \gamma E\vec{\beta} + \vec{k} + (\gamma - 1)(\hat{\beta}\cdot\vec{k})\hat{\beta} \end{pmatrix} = \begin{pmatrix} \gamma E + \gamma\beta k\cos\theta \\ (\gamma\beta E + \gamma k\cos\theta)\hat{\beta} + k\sin\theta\,\hat{n} \end{pmatrix}$$

where $\hat{\beta} \perp \hat{n}$.

Kinematics of 2-body scattering

Suppose the reaction is labeled as b(a,1)2, where a→1, b→2 after scattering. The four momenta of the incident channel are

$$\mathbb{P}_a = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} \\ \overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}_b = \begin{pmatrix} m_b \\ \vec{0} \end{pmatrix}$$

The center of mass 4-vector is

$$\mathbb{P}_c = \mathbb{P}_a + \mathbb{P}_b = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} + m_b \\ \overrightarrow{k_a} \end{pmatrix} = \begin{pmatrix} E_c \\ \overrightarrow{k_a} \end{pmatrix}$$

The system mass is $M_c = \sqrt{E_c^2 - k_a^2} = \sqrt{m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2}}$

The Lorentz boost vector is

$$\vec{\beta} = \frac{\overrightarrow{k_a}}{E_c}, \qquad \gamma = \frac{E_c}{M_c}, \qquad \gamma\vec{\beta} = \frac{\overrightarrow{k_a}}{M_c}$$

The system undergoes a Lorentz boot, so that the total momentum from $\vec{0}$ to $\overrightarrow{k_a} = \gamma\vec{\beta}M_c$, to see that, in the CM frame,

$$\mathbb{P}'_c = \begin{pmatrix} \gamma E_c - \gamma\vec{\beta} \cdot \overrightarrow{k_a} \\ -\gamma\vec{\beta}E_c + \gamma\overrightarrow{k_a} \end{pmatrix} = \begin{pmatrix} M_c \\ \vec{0} \end{pmatrix}$$

where

$$\gamma E_c - \gamma\vec{\beta} \cdot \overrightarrow{k_a} = \gamma E_c - \gamma\frac{k_a^2}{E_c} = \frac{\gamma}{E_c}(M_c^2) = M_c$$

In CM frame,

$$\mathbb{P}'_a = \begin{pmatrix} \gamma\sqrt{m_a^2 + k_a^2} - \gamma\vec{\beta} \cdot \overrightarrow{k_a} \\ -\gamma\vec{\beta}\sqrt{m_a^2 + k_a^2} + \gamma\overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}'_b = \begin{pmatrix} \gamma m_b \\ -\gamma\vec{\beta}m_b \end{pmatrix}$$

Check the energy part

$$\gamma\sqrt{m_a^2 + k_a^2} - \gamma\vec{\beta} \cdot \overrightarrow{k_a} + \gamma m_b = \gamma E_c - \gamma\vec{\beta} \cdot \overrightarrow{k_a} = M_c$$

The momentum part

$$-\gamma\vec{\beta}\sqrt{m_a^2 + k_a^2} + \gamma\overrightarrow{k_a} - \gamma\vec{\beta}m_b = \gamma\vec{\beta}E_c + \gamma\overrightarrow{k_a} = 0$$

After the scattering, only direction changed.

$$p^2 = \frac{1}{4M_c^2}(M_c^2 - (m_1 + m_2)^2)(M_c^2 - (m_1 - m_2)^2)$$

The 4-momenta

$$\mathbb{P}'_1 = \begin{pmatrix} \sqrt{m_1^2 + p^2} \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}'_2 = \begin{pmatrix} \sqrt{m_2^2 + p^2} \\ -\vec{p} \end{pmatrix}$$

Return to the Lab frame

$$\mathbb{P}_1' = \begin{pmatrix} \gamma\sqrt{m_1^2 + p^2} + \gamma\vec{\beta}\cdot\vec{p} \\ \gamma\sqrt{m_1^2 + p^2}\vec{\beta} + \vec{p} + (\gamma - 1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}, \qquad \mathbb{P}_2' = \begin{pmatrix} \gamma\sqrt{m_2^2 + p^2} - \gamma\vec{\beta}\cdot\vec{p} \\ \gamma\sqrt{m_2^2 + p^2}\vec{\beta} - \vec{p} - (\gamma - 1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}$$

The momentum part can be rewritten using $\hat{\beta}\cdot\vec{p} = p\ \cos\theta$

$$\mathbb{P}_1' = \begin{pmatrix} \gamma\sqrt{m_1^2 + p^2} + \gamma\beta p\cos\theta \\ \left(\gamma\beta\sqrt{m_1^2 + p^2} + \gamma p\cos\theta\right)\hat{\beta} + p\sin\theta\ \hat{n} \end{pmatrix}, \mathbb{P}_2' = \begin{pmatrix} \gamma\sqrt{m_2^2 + p^2} - \gamma\beta p\cos\theta \\ \left(\gamma\beta\sqrt{m_2^2 + p^2} - \gamma p\cos\theta\right)\hat{\beta} - p\sin\theta\ \hat{n} \end{pmatrix}$$

where $\hat{\beta} \perp \hat{n}$.

The total energy must be conserved,

$$E_1 + E_2 = \gamma\sqrt{m_1^2 + p^2} + \gamma\sqrt{m_2^2 + p^2} = \sqrt{m_a^2 + k_a^2} + m_b$$

The opening angle

$$k_1 k_2 \cos\theta_{12} = \left(\gamma\beta\sqrt{m_1^2 + p^2} + \gamma p\cos\theta\right)\left(\gamma\beta\sqrt{m_2^2 + p^2} - \gamma p\cos\theta\right) - p^2\sin^2\theta$$

Assume the masses of 1,2 are equal the masses of a, b

After the scattering, only direction changed. Also, we can check the momentum formula

$$p^2 = \frac{1}{4M_c^2}(M_c^2 - (m_a + m_b)^2)(M_c^2 - (m_a - m_b)^2) \overset{?}{\rightarrow} \gamma^2\beta^2 m_b^2$$

Each term,

$$M_c^2 - (m_a + m_b)^2 = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - (m_a + m_b)^2 = 2m_b\sqrt{m_a^2 + k_a^2} - 2m_a m_b$$

$$M_c^2 - (m_a - m_b)^2 = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - (m_a - m_b)^2 = 2m_b\sqrt{m_a^2 + k_a^2} + 2m_a m_b$$

$$p^2 = \frac{1}{4M_c^2}\left(4m_b^2(m_a^2 + k_a^2) - 4m_a^2 m_b^2\right) = m_b^2\frac{k_a^2}{M_c^2} = \gamma^2\beta^2 m_b^2$$

Assume the mass of a and b are the same

Suppose the masses $m_a = m_b = m = m_1 = m_2$

$$\mathbb{P}'_a = \begin{pmatrix} \gamma\sqrt{m^2+k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ -\gamma\vec{\beta}\sqrt{m^2+k_a^2} + \gamma\overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}'_b = \begin{pmatrix} \gamma m \\ -\gamma\vec{\beta}m \end{pmatrix}$$

$$p^2 = \frac{M_c^2}{4} - m^2 = \gamma^2\beta^2 m^2 \Longrightarrow M_c^2 = 4(\gamma^2\beta^2+1)m^2 = 4\gamma^2 m^2$$

$$M_c = 2\gamma m$$

As we expected, as the particle a and b should share equal energy in CM frame, i.e.

$$\gamma\sqrt{m^2+k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} = \gamma m$$

Which can be obtained using

$$m^2 = \left(\gamma\sqrt{m^2+k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a}\right)^2 - \gamma^2\beta^2 m^2$$

The scattered 4-momenta in CM frame are

$$\mathbb{P}'_1 = \begin{pmatrix} \gamma m \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}'_2 = \begin{pmatrix} \gamma m \\ -\vec{p} \end{pmatrix}$$

In Lab frame,

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m + \gamma\vec{\beta}\cdot\vec{p} \\ \gamma^2\vec{\beta}m + \vec{p} + (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \gamma^2 m - \gamma\vec{\beta}\cdot\vec{p} \\ \gamma^2\vec{\beta}m - \vec{p} - (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}$$

The momentum part can be rewritten using $\hat{\beta}\cdot\vec{p} = p\ \cos\theta = \gamma\beta m\cos\theta$

$$\gamma^2\vec{\beta}m + \vec{p} + (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} = (\gamma^2\beta m + \gamma p\cos\theta)\hat{\beta} + p\sin\theta\ \hat{n}$$

Where $\hat{\beta} \perp \hat{n}$.

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m(1+\beta^2\cos\theta) \\ \gamma^2\beta m(1+\cos\theta)\hat{\beta} + p\sin\theta\ \hat{n} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \gamma^2 m(1-\beta^2\cos\theta) \\ \gamma^2\beta m(1-\cos\theta)\hat{\beta} - p\sin\theta\ \hat{n} \end{pmatrix}$$

When scattering angle $\theta = 0$

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m(1+\beta^2) \\ 2\gamma^2 m\,\vec{\beta} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$$

Check:

$$\gamma^2 m(1+\beta^2) = \gamma^2 m + (\gamma^2-1)m = 2\gamma^2 m - m = \gamma M_c - m = E_c - m = \sqrt{m^2+k_a^2}$$

$$2\gamma^2 m\,\vec{\beta} = 2\gamma m\frac{\overrightarrow{k_a}}{M_c} = \overrightarrow{k_a}$$

The total energy or energy conservation,

$$E_1 + E_2 = \gamma^2 m(1+\beta^2\cos\theta) + \gamma^2 m(1-\beta^2\cos\theta) = 2\gamma^2 m = E_c$$

The opening angle

$$k_1 k_2 \cos\theta_{12} = \gamma^4\beta^4 m^2 \sin^2\theta$$

Assume the mass of 1 and 2 becomes equal after scattering

Suppose the mass becomes m, the momentum is

$$p^2 = \frac{M_c^2}{4} - m^2 = \frac{1}{4}\left(m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - 4m^2\right)$$

$$\mathbb{P}_1' = \begin{pmatrix} \frac{M_c}{2} \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}_2' = \begin{pmatrix} \frac{M_c}{2} \\ -\vec{p} \end{pmatrix}$$

$$\mathbb{P}_1 = \begin{pmatrix} \frac{\gamma M_c}{2} + \gamma\beta p\cos\theta \\ \left(\frac{\gamma\beta M_c}{2} + \gamma p\cos\theta\right)\hat{\beta} + p\sin\theta\,\hat{n} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \frac{\gamma M_c}{2} - \gamma\beta p\cos\theta \\ \left(\frac{\gamma\beta M_c}{2} - \gamma p\cos\theta\right)\hat{\beta} - p\sin\theta\,\hat{n} \end{pmatrix}$$

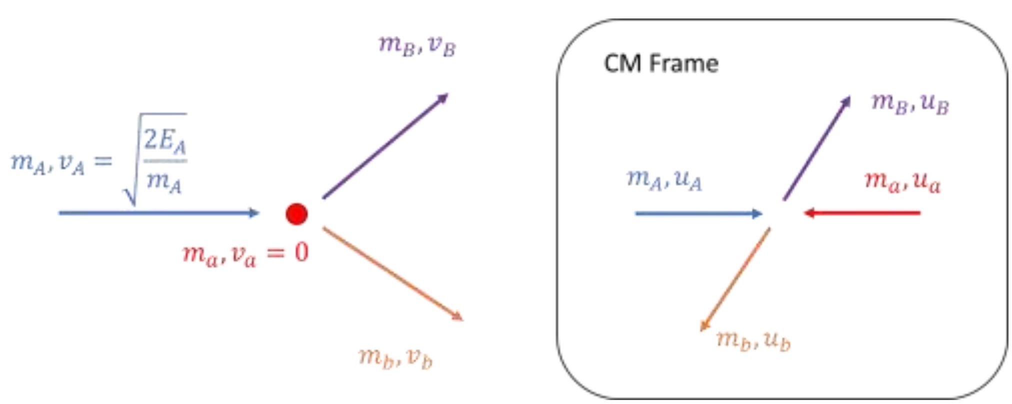
Opening angle

$$k_1 k_2 \cos\theta_{12} = \frac{\gamma^2\beta^2 M_c^2}{4} - \gamma^2\beta^2 p^2\cos^2\theta - p^2$$

$$= \frac{\gamma^2\beta^2 M_c^2}{4} - \gamma^2\beta^2 p^2\cos^2\theta - p^2 = \frac{\gamma^2\beta^2 M_c^2}{4} - \frac{M_c^2}{4} + m^2 - \gamma^2\beta^2 p^2\cos^2\theta$$

Non-relativistic Transfer reaction

The reaction in the Lab frame is A(a,b)B, the velocity in the Lab frame is donated by v , in the CM frame, the velocity denotes by u.



The velocity of particle A and the kinetic energy E_A is related by $E_A = \frac{mv_A^2}{2}$, or $v_A = \sqrt{2E_A/m_A}$. The velocity of the CM (center-of-mass) frame is

$$V_{cm} = \frac{m_A}{m_A + m_a} v_A = \frac{\sqrt{2m_A E_A}}{m_A + m_a}$$

The particle's velocities in the CM frame are

$$u_A = v_A - V_{cm} = \frac{m_a}{m_A} V_{cm}$$
$$u_a = v_a - V_{cm} = -V_{cm}$$

The total momentum

$$m_A u_A + m_a u_a = 0$$

The total kinetic energy in the CM frame

$$T_{cm} = \frac{1}{2} m_A u_A^2 + \frac{1}{2} m_a u_a^2 = \frac{1}{2} \frac{m_a}{m_A} (m_A + m_a) V_{cm}^2 = \frac{1}{2} \frac{m_a m_A}{m_a + m_A} v_A^2 = \frac{m_a}{m_A + m_a} E_A$$

After scattering, the balance of energy and momentum are

$$\frac{1}{2} m_b u_b^2 + \frac{1}{2} m_B u_B^2 = T_{cm} + Q, \qquad Q = m_A + m_a - m_b - m_B$$
$$m_b u_b + m_B u_B = 0$$

The solution for the particle velocities is

$$u_b = \sqrt{\frac{m_B}{m_b}} \sqrt{\frac{2(Q + T_{cm})}{m_b + m_B}}, \qquad u_B = \sqrt{\frac{m_b}{m_B}} \sqrt{\frac{2(Q + T_{cm})}{m_b + m_B}}$$

Move back to the Lab frame,

$$\overrightarrow{v_b} = \begin{pmatrix} -u_b \cos\theta_{cm} + V_{cm} \\ -u_b \sin\theta_{cm} \end{pmatrix}, \qquad \overrightarrow{v_B} = \begin{pmatrix} u_B \cos\theta_{cm} + V_{cm} \\ u_B \sin\theta_{cm} \end{pmatrix}$$

Let's focus on the light particle form now on, it carries charge Z and moves in a magnetic field strength B in the z-direction. The distance in the z-direction after time t_c is

$$\frac{z}{t_c} = -u_b \cos\theta_{cm} + V_{cm}$$

The kinetic energy is

$$E_b = \frac{1}{2} m_b \left(u_b^2 + V_{cm}^2 - 2u_b V_{cm} \cos\theta_{cm}\right)$$

Eliminate θ_{cm} both equations, we have

$$E_b = \frac{1}{2} m_b u_b^2 - \frac{1}{2} m_b V_{cm}^2 + \frac{m_b V_{cm}}{t_c} z$$

The excitation energy of particle B is implicitly in u_b, and all other coefficients are fixed.

The kinetic energy of particle b in CM frame is

$$T_b = \frac{1}{2} m_b\, u_b^2 = \frac{m_B(Q + T_{cm})}{m_b + m_B} \Longrightarrow Q + T_{cm} = \frac{m_b + m_B}{m_B} T_b$$

Thus, the Ex-spectrum of particle B is, substitute $Q \to\ Q_0 - E_x$,

$$E_x = Q_0 + T_{cm} - \frac{m_b + m_B}{m_B}\left(E_b + \frac{1}{2} m_b V_{cm}^2 - \frac{m_b V_{cm}}{t_c} z\right)$$

Projection of Ex for different reactions with same light particle charge state

Suppose we have A(p,d), A(d,d) and A(d,t) reactions happened all at the same experiment, Since the charge states of all light recoil are the same, and the slope of the e-z plot only depends on the charge state, so that all reactions will have same slope and all reaction can be projected.

The classical limit is

$$E_b = T_b - \frac{1}{2} m_b V_{cm}^2 + \eta z, \qquad \eta = \frac{m_b V_{cm}}{t_c}$$

And the Q-value

$$Q = \frac{m_b + m_B}{m_B} T_b - T_{cm}$$

The excitation energy in inside the Q-value, substitute $Q \to Q_0 - E_x$

$$Q_0 - E_x = \frac{m_b + m_B}{m_B} T_b - T_{cm}$$

$$E_x = Q_0 + T_{cm} - \frac{m_b + m_B}{m_B} T_b$$

$$E_x = F - \frac{m_b + m_B}{m_B}(E_b - \eta z), \qquad F = Q_0 + T_{cm} - \frac{m_b + m_B}{2} \frac{m_b}{m_B} V_{cm}^2$$

We can see, the projection $E_b - \eta z$ need a scaling factor $(m_b + m_B)/m_B$ to give the correct scale. And there are a few additional terms to do the offset.

For mass of particle-A is larger than deuteron and fixed beam energy, the term

$$V_{cm} = \frac{\sqrt{2 m_A E_A}}{m_A + m_a} = \frac{\sqrt{2E_A/m_A}}{1 + m_a/m_A} \approx \sqrt{\frac{2E_A}{m_A}} = const.$$

$$T_{cm} = \frac{m_a}{m_A + m_a} E_A \approx \frac{m_a}{m_A} E_A$$

So, the offset factor approximately equal to

$$F \approx Q_0 + \frac{(m_a - m_b) m_B - m_b^2}{m_A m_B} E_A$$

Replace all mass with mass number times the nucleon mass m_u and simplify,

$$F \approx Q_0 + \frac{(a-b)B - b^2}{AB} E_A$$

For the A(p,p), A(p,d), A(d,p), A(d,d) and A(d,t) reactions,

$$\begin{aligned}
F(p,p) &\approx -\frac{1}{AB} E_A \\
F(p,d) &\approx Q_0(p,d) - \frac{B-4}{AB} E_A \\
F(d,p) &\approx Q_0(d,p) + \frac{1}{B} E_A \\
F(d,d) &\approx -\frac{4}{AB} E_A \\
F(d,t) &\approx Q_0(d,t) - \frac{B+9}{AB} E_A
\end{aligned}$$

They have different offsets.