Kinematics in HELIOS detector and particle detection

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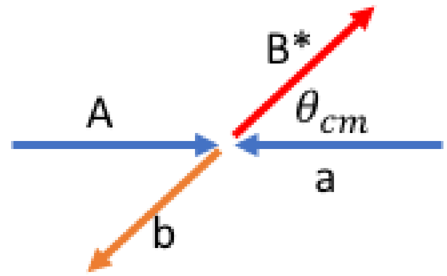
Contents

Transfer Reaction

The kinematics of transfer reaction, denote as A(a,b)B, where A is incoming particle with larger mass, a is target nucleus, b and B are scattered particles, in which b is the lighter one.

The four-momentum vector of particle b and B is



$$\mathbb{P}_b = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \gamma q - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta q - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E \\ p\cos\theta \\ p\sin\theta \end{pmatrix}$$

$$\mathbb{P}_B = \begin{pmatrix} E' \\ p'_z \\ p'_{xy} \end{pmatrix} = \begin{pmatrix} \gamma Q + \gamma\beta k \cos\theta_{cm} \\ \gamma\beta Q + \gamma k \cos\theta_{cm} \\ -k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E' \\ p'\cos\theta \\ p'\sin\theta \end{pmatrix}$$

Where

$$q = \sqrt{m_b^2 + k^2} = \frac{1}{2E_t}\left(E_t^2 + m_b^2 - m_B^2\right)$$

$$Q = \sqrt{m_B^2 + k^2} = \frac{1}{2E_t}\left(E_t^2 - m_b^2 + m_B^2\right)$$

$$k^2 = \frac{1}{4E_t^2}\left(E_t^2 - (m_b + m_B)^2\right)\left(E_t^2 - (m_b - m_B)^2\right)$$

$$M_c^2 = E_t^2 = 2m_a(m_A + T) + m_a^2 + m_A^2 = (m_a + m_A)^2 + 2m_a T$$

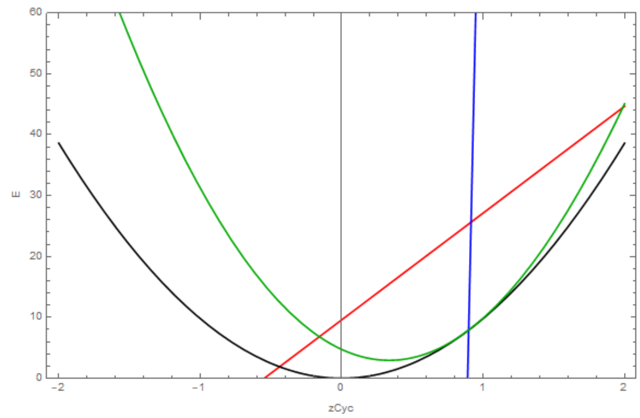
$$\beta = \frac{\sqrt{(m_A + T)^2 - m_A^2}}{m_a + m_A + T}, \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

In here θ_{cm} is the center of mass scattering angle. k is the momentum of particle b or B in CM frame. E_t is the total energy in CM frame, or the mass of the system M_c. q is the total energy of particle b in CM frame. T is the total kinetic energy of particle A.

$$p = \frac{\frac{\gamma}{\cos\theta}}{1 + \gamma^2 \tan^2\theta}\left(q\beta + \sqrt{k^2 + (k^2 - q^2\beta^2)\gamma^2 \tan^2\theta}\right)$$

$$\tan\theta_{cm} = \frac{p\sin\theta}{\beta q - \frac{p}{\gamma}\cos\theta}$$

Using this four-momentum vector, we are going to give out the formula that use in HELIOS. The most representation plot is the e – z plot, where the kinetic energy versus position along the HELIOS axis. A typical plot like this:



The black curve is the lower or upper bound of the energy. The red line the locus for fixed excitation energy. The blue line is the line for $\theta_{cm} = 0$, and the green curve is constant θ_{cm} or $\pi - \theta_{cm}$.

The basic formula is the rotation radius

$$\rho = \frac{P}{cZB}$$

where P is momentum in MeV/c that perpendicular to the magnetic field B (in T), Z is the charge state, $c = 299.792458$. the unit of ρ is meter. Under the kinematics of transfer reaction

$$\rho = \frac{p_{xy}}{cZB} = \frac{k \sin\theta_{cm}}{cZB}$$

The time for a cycle is

$$t = \frac{2\pi\rho}{v_\perp} = \frac{2\pi}{cZB}\frac{k\sin\theta_{cm}}{v_\perp}$$

The time for a cycle is almost fixed. Thus the length for a cycle is

$$z_0 = v_\parallel t = 2\pi\rho\frac{v_\parallel}{v_\perp} = \frac{2\pi}{cZB}\frac{v_\parallel}{v_\perp}k\sin\theta_{cm}, \qquad \frac{v_\parallel}{v_\perp} = \frac{1}{\tan\theta}$$

$$z_0 = 2\pi\frac{\rho}{\tan\theta} = \frac{2\pi}{cZB}p_z$$

$$\alpha\, z = p_z = \gamma\beta q - \gamma k\cos\theta_{cm}, \qquad \alpha = \frac{cZB}{2\pi}$$

With the energy equation, we have 2 coupled equations:

$$\alpha\, z = \gamma\beta q - \gamma k\cos\theta_{cm}$$
$$e + m_b = \gamma q - \gamma\beta k\cos\theta_{cm}$$

By eliminate difference variable, we can get all difference curves or line.

The constant E_x line

First, by eliminating $\cos\theta_{cm}$, we get the red line, which only depends on excitation energy

$$e = \frac{1}{\gamma}q - m_b + \alpha\beta z = \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma E_t} - m_b + \frac{cZB}{2\pi}\beta z$$

The intercept of the red line is

$$e_0 = \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma E_t} - m_b$$

The only non-constant is m_B, which can be excited. Let examine the term, for $E_x \ll m_B$

$$\frac{m_B^2}{2\gamma E_t} \to \frac{(m_B + E_x)^2}{2\gamma E_t} \approx \frac{m_B^2}{2\gamma E_t}\left(1 + \frac{2E_x}{m_B}\right) = \frac{m_B^2}{2\gamma E_t} + \frac{m_B}{\gamma E_t}E_x$$

At small incident energy, $M_c = m_b + m_B + T_{cm} \approx m_B, \gamma \approx 1$,

$$e_0 \approx \frac{M_c^2 + m_b^2 - m_B^2}{2\gamma E_t} - m_b - E_x$$

second, we can also eliminate e, so that,

$$\cos\theta_{cm} = \frac{\beta q}{k} - \frac{\alpha}{\gamma k}z$$

This is the center-of-mass-angle to z-position relationship. The dependency of the excitation energy is inside the term q.

The constant θ_{cm} line

Next, we eliminate m_B, notice that $k = k(m_B)$, we need to eliminate that as well, we have a complicated curve

$$e = -m_b + \frac{-\sin^2(\theta_{cm})\,\alpha\beta\gamma^2 z + \cos\theta_{cm}\sqrt{\alpha^2 z^2 + m_b^2(1 - \sin^2(\theta_{cm})\,\gamma^2)}}{1 - \sin^2(\theta_{cm})\,\gamma^2}$$

This is a general contour for a given θ_{cm}. When $\theta_{cm} = 0$, it reduces to

$$e = -m_b + \sqrt{\alpha^2 z^2 + m_b^2}$$

This is the black curve. When $\theta_{cm} = \frac{\pi}{2}$,

$$e = -m_b + \frac{\alpha}{\beta}z$$

This is the blue line.

The Bore radius line

Since the detector may have maximum radius R, and $2\rho \leq R$. Thus,

$$\rho = \frac{k \sin \theta_{cm}}{cZB} \leq \frac{R}{2} \Rightarrow k \sin \theta_0 = R \frac{cZB}{2} = R\alpha\pi$$

Put in the 2 coupled equations:

$$\begin{aligned} \alpha\, z &= \gamma\beta q - \gamma\sqrt{k^2 - (R\alpha\pi)^2} \\ e + m_b &= \gamma q - \gamma\beta\sqrt{k^2 - (R\alpha\pi)^2} \end{aligned}$$

Expand in recoil mass m_B,

$$k^2 = \frac{1}{4E_t^2}\left(E_t^2 - (m_b + m_B)^2\right)\left(E_t^2 - (m_b - m_B)^2\right)$$

$$\alpha\, z = \gamma\beta \frac{1}{2E_t}\left(E_t^2 + m_b^2 - m_B^2\right) - \gamma\sqrt{\frac{1}{4E_t^2}\left(E_t^2 - (m_b + m_B)^2\right)\left(E_t^2 - (m_b - m_B)^2\right) - (R\alpha\pi)^2}$$

$$e + m_b = \gamma \frac{1}{2E_t}\left(E_t^2 + m_b^2 - m_B^2\right) - \gamma\beta\sqrt{\frac{1}{4E_t^2}\left(E_t^2 - (m_b + m_B)^2\right)\left(E_t^2 - (m_b - m_B)^2\right) - (R\alpha\pi)^2}$$

Eliminate m_B

$$2m_b e + e^2 = \alpha^2(\pi^2 R^2 + z^2)$$

$$e = \sqrt{\alpha^2(\pi^2 R^2 + z^2) + m_b^2} - m_b$$

Compare with the constant $\theta_{cm} = 0$ line

$$e = \sqrt{\alpha^2 z^2 + m_b^2} - m_b^2$$

The

Maximum excitation energy

We can see that, when the excitation energy of particle B is higher, the red line shifts lower, there is an upper limit for the red line to be shifted, which is when the red line touches the black curve.

$$(e_{max}, z_{max}) = \left(m_b\gamma - m_b, \frac{\gamma\beta}{\alpha} m_b\right)$$

Solve for the maximum m_B'

$$m_B(\max) = M_c - m_b = \sqrt{(m_a + m_A)^2 + 2m_a T} - m_b$$

Which make perfect sense in CM frame. At non-relativistic limit,

$$m_B(\max) = \sqrt{(m_a+m_A)^2 + 2m_aT} - m_b \rightarrow m_a + m_A + \frac{m_aT}{m_a+m_A} - m_b = Q + m_B + \frac{m_aT}{m_a+m_A}$$

$$E_x(\max) = Q + \frac{m_aT}{m_a+m_A} = Q + T_{cm}$$

Where T_{cm} is the CM frame kinematics energy.

Minimum Incident energy

The minimum incident energy requires that

$$M_c \geq m_b + m_B \Rightarrow (m_a+m_A)^2 + 2m_aT_{min} = (m_b+m_B)^2$$

$$T_{min} = \frac{(m_b+m_B)^2 - (m_a+m_A)^2}{2m_a} \cong -Q\left(1+\frac{m_A}{m_a}\right) \neq Q$$

Tilted Reaction

When the incident particle with some incident angle θ_A, the four-momentum of particle b will be tilted by angle θ_A,

$$\mathbb{P}_b = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_A & -\sin\theta_A \\ 0 & \sin\theta_A & \cos\theta_A \end{pmatrix} \begin{pmatrix} \gamma q - \gamma\beta k\cos\theta_{cm} \\ \gamma\beta q - \gamma k\cos\theta_{cm} \\ k\sin\theta_{cm} \end{pmatrix}$$

Since the z-position is

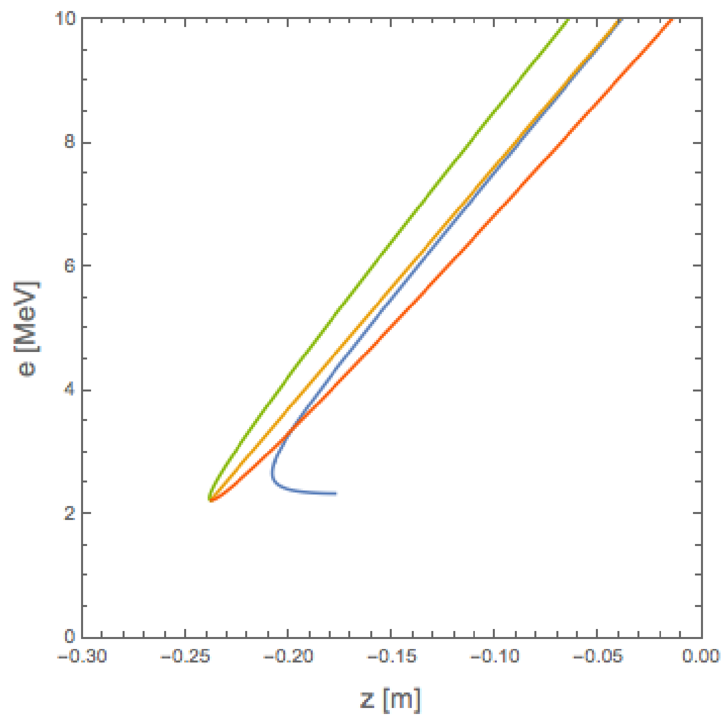
$$\alpha\, z = p_z = (\gamma\beta q - \gamma k\cos\theta_{cm})\cos\theta_A + k\sin(\theta_{cm})\sin\theta_A$$

With the energy

$$e + m = q - \gamma\beta k\cos\theta_{cm}$$

Eliminate θ_{cm}, we got

$$\alpha\beta z = \left(e + m - \frac{q}{\gamma}\right)\cos\theta_A + \frac{1}{\gamma}\sqrt{(\gamma\beta k)^2 - (q-e-m)^2}\sin\theta_A$$



In the above plot, the orange line is the normal constant E_x line, the green curve is $\theta_A = 50$ mrad, and the red curve is $\theta_A = -50$ mrad. 50 mard ~ 2.9 deg. And the blue curve is with finite detector correction, such that $a = 0.01$ meter. The reaction is 208Pb(d,p) at 10 MeV/u, magnetic field is 2.85 T.

In this calculation, we can see the finite emittance of the beam could contribute a lot to the energy resolution.

Alpha Source

When alpha source is put at the axis, the 4-momentum is

$$\mathbb{P} = (E, p_z, p_{xy}), \qquad E = m_\alpha + T, \qquad p_z = p\cos\theta\,, \qquad p = \sqrt{2m_\alpha T + T^2}$$

Under a magnetic field, the bending radius is

$$\rho = \frac{P}{cZB}$$

where P is momentum in MeV/c that perpendicular to the magnetic field B (in T), Z is the charge state, $c = 299.792458$. The unit of ρ is meter

$$\rho = \frac{p\sin\theta}{cZB},$$

The time for a cycle is

$$t = \frac{2\pi\rho}{v_\perp} = \frac{2\pi}{cZB}\frac{p\sin\theta}{v_\perp}$$

Thus the length for a cycle is

$$z_0 = v_\parallel t = 2\pi\rho \frac{v_\parallel}{v_\perp} = \frac{2\pi}{cZB}\frac{v_\parallel}{v_\perp} p \sin\theta = \frac{2\pi}{cZB} p\cos\theta\,, \qquad \frac{v_\parallel}{v_\perp} = \frac{1}{\tan\theta}$$

$$z_0 = \frac{2\pi}{cZB} p\cos\theta$$

The locus is

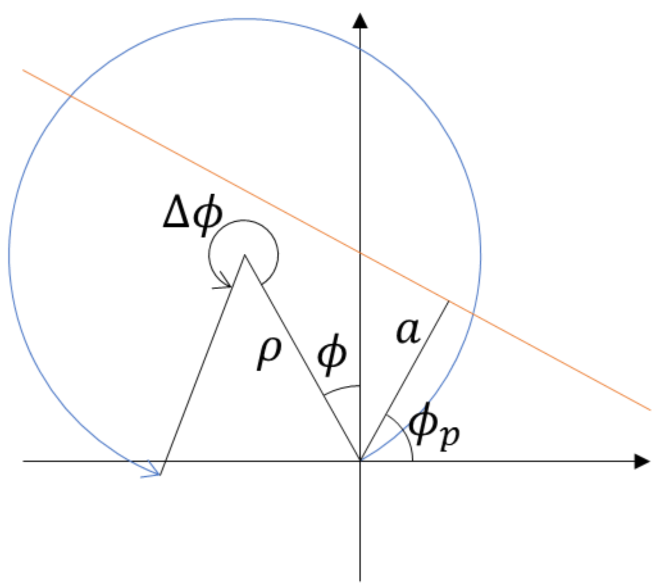
$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} \sin\phi - \sin\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \end{pmatrix}$$

The radius is

$$r = \sqrt{x^2 + y^2} = \sqrt{2}\rho\sqrt{1 - \cos\left(\tan(\theta)\frac{z}{\rho}\right)}$$

Finite axial detector

A finite axial detector is a polygonal prism that surrounded and centered the HELIOS axis and larger than the beam size. The blue circle is the XY projection of the particle trajectory. The orange line is one of the detector plan.



For am axial detector, the normal of a plane is

$$\hat{n} = (\cos\phi_p\,, \sin\phi_p\,, 0)$$

The equation for the detector plane is

$$x\cos\phi_p + y\sin\phi_p = a$$

Where a is the shortest distance from the plane to z-axis.

The equation of the locus of the positive charged particle, when the B-field is direct to the z-axis, is

$$\binom{x}{y} = \rho \begin{pmatrix} \sin\phi - \sin\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \end{pmatrix}$$

Where (θ, ϕ) is the scattering angle of particle b.

The hit points are

$$z_{hit} = \frac{\rho}{\tan(\theta)}\left(\phi_p - \phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)\right), n = 0,1,2 \ldots.$$

For real solution,

$$-1 < \frac{a}{\rho} + \sin(\phi - \phi_p) < 1$$

Notice that, the length for a cycle is

$$z_0 = \frac{2\pi\,\rho}{\tan(\theta)}$$

$$z_{hit} = \frac{z_0}{2\pi}\left(\phi_p - \phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)\right)$$

Since we want the know which hit-point is hit from outside, i.e. the direction of the particle is toward the axis, not outward from the axis.

The direction vector for the particle is

$$\frac{d}{dz}\binom{x}{y} = \rho\tan(\theta) \begin{pmatrix} \cos\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \\ \sin\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \end{pmatrix}$$

The dot product with the plane normal

$$\cos\theta' = \rho\tan(\theta)\cos\left(\tan(\theta)\frac{z}{\rho} + \phi - \phi_p\right) < 0$$

$$\Rightarrow \cos\left(\tan(\theta)\frac{z}{\rho} + \phi - \phi_p\right) < 0$$

In fact, using geometrical argument, for $n = odd$ number, the hit point is always inward. Substitute z_{hit}

$$\begin{aligned} &\mathrm{Cos}\left(n\pi + (-1)^n \sin^{-1}\left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)\right) \\ &= (-1)^n \cos\left(\sin^{-1}\left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)\right) \end{aligned}$$

$$= (-1)^n \sqrt{1 - \left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)^2} < 0$$

This prove the geometrical argument that $n = odd$.

A special case for $\phi = 0, \phi_p = \pi, n = 1$

$$z_{hit} = \frac{\rho}{\tan(\theta)}\left(2\pi - \sin^{-1}\left(\frac{a}{\rho}\right)\right) = z_0\left(1 - \frac{1}{2\pi}\sin^{-1}\left(\frac{a}{\rho}\right)\right)$$

The rotated angle is smaller then 2π. When $\rho \gg a$

$$z_{hit} \approx z_0\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right)$$

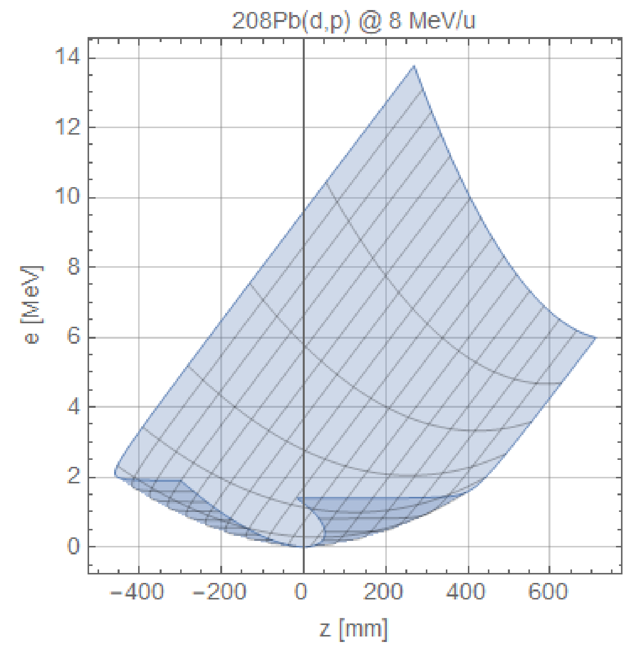
With the transfer reaction

The red line becomes

$$\alpha\beta\, z_{hit} \approx \left((m+e) - \frac{q}{\gamma}\right)\left(1 - \frac{\alpha a}{\sqrt{k^2 - \left(\frac{\gamma q - m - e}{\gamma\beta}\right)^2}}\right)$$

When e become large, the $\sin^{-1}()$ becomes small, and the formula becomes to normal one.

$$\alpha\beta z = \left(y - \frac{q}{\gamma}\right)\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{\gamma^2\beta^2k^2 - (\gamma q - y)^2}}\right)$$



Off-axis Effect

When the helix of particle b is off-axis.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} \sin\left(\tan(\theta)\frac{z}{\rho} + \phi\right) - \sin\phi \\ \cos\phi - \cos\left(\tan(\theta)\frac{z}{\rho} + \phi\right) \end{pmatrix} + \rho_0 \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

The solution for z_{hit} becomes

$$z_{hit} = \frac{\rho}{\tan(\theta)}\left(\phi_p - \phi + n\pi + (-1)^n \sin^{-1}\left(\frac{a - \rho_0\cos(\phi_0 - \phi_p)}{\rho} + \sin(\phi - \phi_p)\right)\right), n = 0,1,2 \ldots.$$

For $n = 1, \phi_p = \pi, \phi = 0$

$$z_{hit} = z_0\left(1 - \frac{1}{2\pi}\sin^{-1}\left(\frac{a + \rho_0\cos(\phi_0)}{\rho}\right)\right), \qquad \left|\frac{a + \rho_0\cos(\phi_0)}{\rho}\right| < 1$$

We can see, the effect is same as change of the effective $a_{eff} = a - \rho_0\cos(\phi_0 - \phi_p)$. Also the beam size must be smaller than the detector distance $\rho > a > \rho_0\cos(\phi_0 - \phi_p)$.

Radial Detector (no conclusion yet)

A radial detector is a plane detector located at fixed z-pos (z_R) and perpendicular to HELIOS axis. The hit position with an off-axis helix is

$$\binom{x}{y} = \rho \begin{pmatrix} \sin\left(\tan(\theta)\frac{z_R}{\rho} + \phi\right) - \sin\phi \\ \cos\phi - \cos\left(\tan(\theta)\frac{z_R}{\rho} + \phi\right) \end{pmatrix} + \rho_0 \binom{\cos\phi_0}{\sin\phi_0}$$

For $\rho_0 = 0$, the radial position is

$$r = \sqrt{x^2 + y^2} = \rho\sqrt{2 - 2\cos\left(\tan(\theta)\frac{z_R}{\rho}\right)}$$

$$\rho = \frac{p_{xy}}{cZB} = \frac{k\sin\theta_{cm}}{cZB}, \qquad \tan\theta = \frac{p_{xy}}{p_z}, \qquad \frac{\tan(\theta)\, z_R}{\rho} = \frac{cZB}{p_z} z_R$$

Since the z-pos is fixed, the TOF from target to the detector is

$$t = \frac{z_R}{v_z} = \frac{z_R}{c}\frac{E}{p_z}$$

Eliminate the $k\cos\theta_{cm}$ in p_z, we have

$$e + m = \frac{q}{\gamma}\frac{ct}{\beta z_R - ct}$$

Knockout reaction

The reaction is notated as A(a, 12)B, where A = B + b, in which b is the bounded nucleus, and 1 and 2 are free scattering particles. When particle b knocked out, it becomes particle 2. The energy and momentum conservation is

$$\mathbb{P}_A + \mathbb{P}_a = \mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_B$$

In which the mass of particle B is

$$m_B + m_2 = m_A + S$$

The reaction Q-value is

$$Q = m_A + m_a - m_1 - m_2 - m_B = -S$$

The recoil of the particle B assumed the form

$$\mathbb{P}_B = \left(m_B, -\overrightarrow{k_b}\right) = \left(\sqrt{(m_A - m_2 + S)^2 + \left|\overrightarrow{k_b}\right|^2}, -\overrightarrow{k_b}\right)$$

Where S is separation energy, $\vec{k}$ is the recoiled momentum, which is same but opposite direction with the bounded nucleus b, as particle A is stationary.

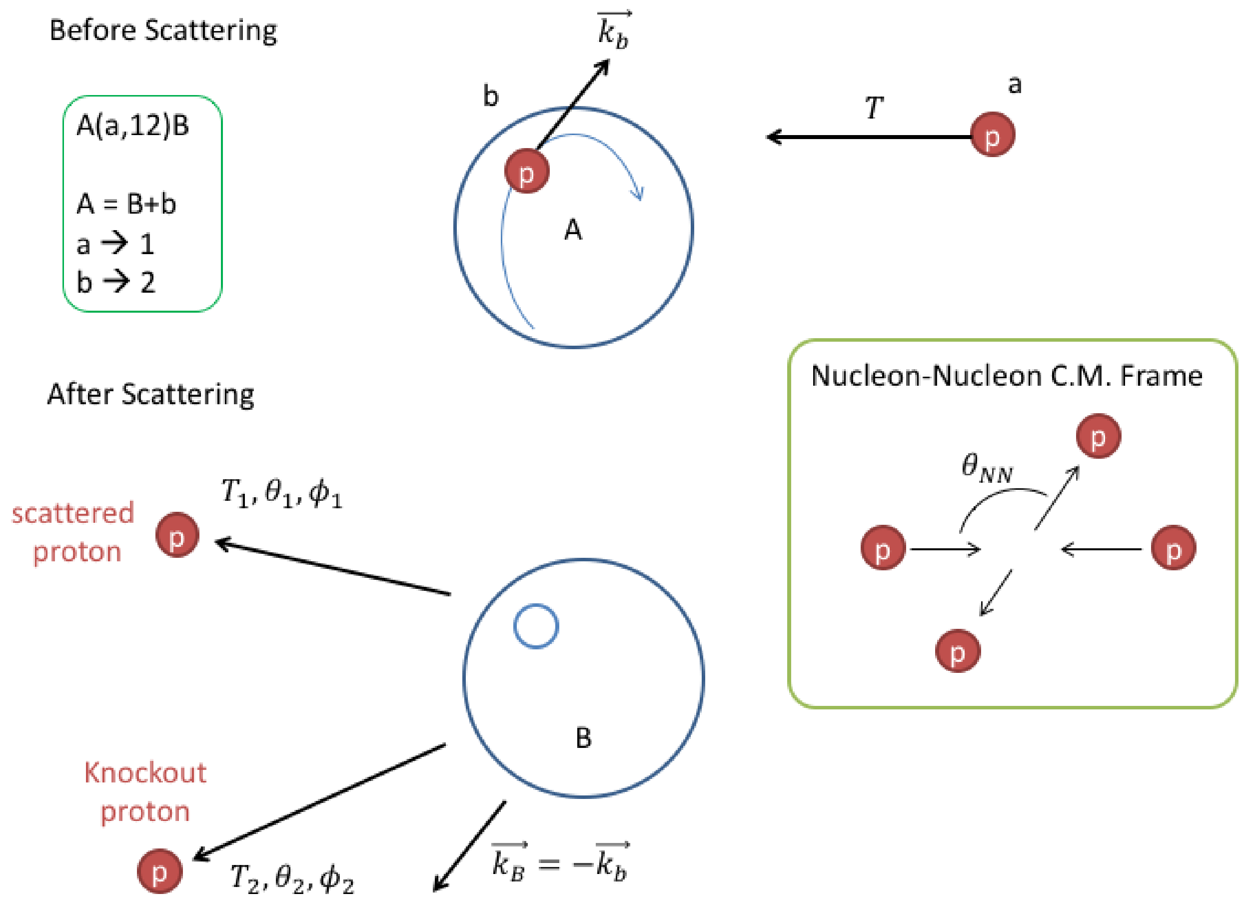
Assume a is the incident particle and A is the target, We can from a quai-particle b by

$$\mathbb{P}_b = \mathbb{P}_A - \mathbb{P}_B$$

$$\mathbb{P}_b = \left(m_b, \vec{k}_b\right) = \left(m_A - \sqrt{(m_A - m_a + S)^2 + \left|\overrightarrow{k_b}\right|^2}, \overrightarrow{k_b}\right)$$

Thus, the rest is similar to that of transfer reaction, except the target is also moving.

Because the "target" b and incident particle a are both moving, this forms the plane of incident channel. And the exit channel, the particle 1 and 2 can be not on the same plane. The following illustration is normal kinematics.



Once the quasi-particle is constructed, the reaction is reduced to

$$\mathbb{P}_a + \mathbb{P}_b = \mathbb{P}_1 + \mathbb{P}_2$$

Thus, the next step of calculation is identical to transfer reaction. The reaction constants are

$$\vec{\beta} = \frac{\overrightarrow{k_a} + \overrightarrow{k_b}}{E_a + E_b}, \qquad \gamma = \frac{1}{\sqrt{1 - |\beta|^2}}$$

$$E_t = \sqrt{(E_a + E_b)^2 + \left|\overrightarrow{k_a} + \overrightarrow{k_b}\right|^2}$$

$$k^2 = \frac{1}{4E_t^2}\left(E_t^2 - (m_1 + m_2)^2\right)\left(E_t^2 - (m_1 - m_2)^2\right)$$

in above, $\vec{\beta}$ is the Lorentz boost to NN-CM frame. E_t is the total energy in NN-CM frame, or the intrinsic total energy of NN-system. k is the magnitude of momentum of the scattered particle 1 and 2 in NN-CM frame.

Since the Lorentz boost of from the Lab frame to the NN-CM (nucleon-nucleon center of mass) frame is not on the z-axis, the formula for the particle 1 and 2 is complicated. In the NN-CM frame, the four-vector for particle 1 is

$$\mathbb{P}_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \sqrt{m_1^2 + k^2} \\ k\cos\theta \\ k\sin\theta \end{pmatrix}$$

Where θ is not the CM frame scattering angle, because the particle a could has some finite polar angle.

Inverse Kinematics

In inverse kinematics, the momentum $\overrightarrow{k_a} = 0$, that simplify the calculation that, the reaction is a tilted transfer reaction, i.e. the reaction axis is not on the z-axis.

Reconstruct scattered four-momentum

In knockout experiment, we need to reconstruct the four momenta. Under HELIOS, to problem is converting z_{hit} to θ_i, the lab angle.

Inverse Problem

We show that the solution from CM frame to Lab frame, or from theory to experiment. Basically, the HELIOS is a problem of finding the mapping

$$\begin{pmatrix} E_x \\ \theta_{cm} \end{pmatrix} \leftrightarrow \begin{pmatrix} e \\ z \end{pmatrix}$$

In term of E_x and $\cos\theta_{cm}$

We can express (z, e) in term of (E_x, θ_{cm}) as

$$\begin{pmatrix} e \\ z \end{pmatrix} = \frac{\gamma}{2E_t}\begin{pmatrix} M_c^2 + m_b^2 - (m_B + E_x)^2 - \beta\cos\theta_{cm}\sqrt{(M_c^2 - (m + m_B + E_x)^2)(M_c^2 - (m - M - E_x)^2)} \\ \beta(M_c^2 + m_b^2 - (m_B + E_x)^2) - \cos\theta_{cm}\sqrt{(M_c^2 - (m + M + E_x)^2)(M_c^2 - (m - M - E_x)^2)} \end{pmatrix}$$

The inverse

$$\begin{pmatrix} E_x \\ \cos\theta_{cm} \end{pmatrix} = \begin{pmatrix} -m_B + \sqrt{M_c^2 + m_b^2 - 2\gamma M_c(E - \alpha\beta z)} \\ \dfrac{\gamma(E\beta - \alpha\, z)}{\sqrt{\gamma^2(E - \alpha\beta z)^2 - m_b^2}} \end{pmatrix}$$

Get k and θ_{cm} from e and z

From experiment, we get the energy ($y = e + m$) and position (z), then we can reconstruct the reaction constant k and θ_{cm}.

$$k^2 = \gamma^2 (y - \beta\alpha z)^2 - m^2$$

$$\cos\theta_{cm} = \frac{(\alpha z - \beta y)}{\gamma k} = \frac{\gamma\sqrt{m^2 + k^2} - y}{\gamma\beta k}$$

From k^2, the total mass of the particle B is

$$m_B^2 = m_b^2 + M_c^2 - 2M_c\sqrt{k^2 + m_b^2}$$

Where M_c is the total mass of the system.

Finite detector

The coupled solution is

$$y = e + m = \gamma\sqrt{m^2 + k^2} - \gamma\beta k \cos\theta_{cm}$$

$$\rho = \frac{k\sin\theta_{cm}}{cZB} = \frac{k\sin\theta_{cm}}{2\pi\alpha}$$

$$\alpha\beta\gamma z = \left(\gamma y - \sqrt{m^2 + k^2}\right)\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right)$$

Solve $\cos\theta$ from 1st equation, and sub into 2nd equation,

$$\alpha\beta\gamma z = \left(\gamma y - \sqrt{m^2 + k^2}\right)\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2y\gamma\sqrt{m^2 + k^2} - y^2 - m^2\gamma^2 - k^2}}\right)$$

Use

$$k \to m\tan(x), \qquad 0 < x < \frac{\pi}{2}$$

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\tan^2(x)}}\right)$$

Under the square root,

$$\begin{aligned}
&2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\sec^2(x) + m^2 \\
&= -y^2\gamma^2 + 2y\gamma m\sec(x) - m^2\sec^2(x) + y^2\gamma^2 - y^2 - m^2\gamma^2 + m^2 \\
&= -(y\gamma - m\sec(x))^2 + (y^2 - m^2)\gamma^2\beta^2
\end{aligned}$$

Than

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{(y^2 - m^2)\gamma^2\beta^2 - (y\gamma - m\sec(x))^2}}\right)$$

Replace

$$\begin{aligned} &y\gamma - m\sec(x) \to K \\ &(y^2 - m^2)\gamma^2\beta^2 \to H^2 > 0 \\ &\alpha\beta\gamma z \to Z \\ &\beta\gamma\alpha a \to G > 0 \end{aligned}$$

$$Z = K\left(1 - \frac{G}{\sqrt{H^2 - K^2}}\right)$$

Next, replace

$$K \to H\sin\phi\,, \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

$$Z = H\sin\phi\left(1 - \frac{G}{H\cos\phi}\right)$$

or

$$Z = H\sin\phi - G\tan\phi$$

The momentum square is

$$k^2 = (\gamma g - H\sin\phi)^2 - m^2$$

When $a \to 0, G \to 0$

$$\begin{aligned} &Z = H\sin\phi = K = \gamma y - m\sec(x) \\ &\to\ \alpha\beta\gamma z = \gamma y - \sqrt{m^2 + k^2} \\ &\to y = \frac{1}{\gamma}\sqrt{m^2 + k^2} + \alpha\beta z \end{aligned}$$

Or

$$k^2 = \gamma^2(y - \alpha\beta z)^2 - m^2$$

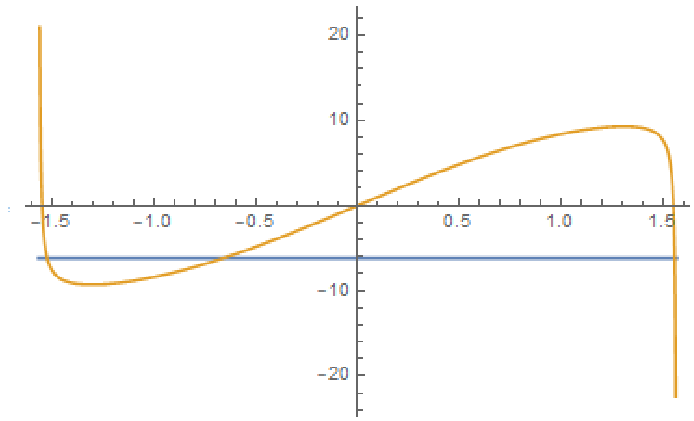
Return to the infinite detector solution.

Since $H, G > 0$, and $G < H$, as the term $\frac{G}{\sqrt{H^2 - K^2}} = \frac{a}{2\pi\rho} < 1$

The function

$$f(\phi) = H\sin\phi - G\tan\ \phi$$

Looks like this

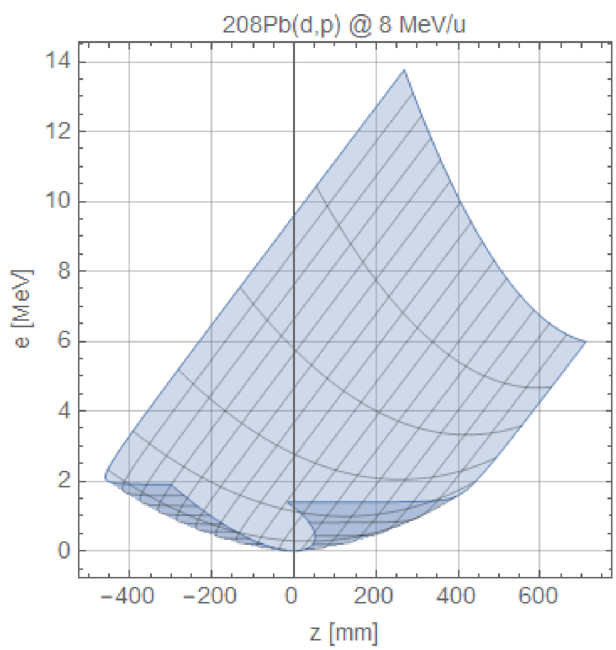


Where the orange line is the $f(\phi)$ and the blue line is $f(\phi) = Z$. We can see, there are multi-solution for ϕ. Normally, when $\theta_{cm} \gg 0$, the derivative is

$$f'(\phi) = H\cos\phi - G\sec^2\phi > 0$$

From experience (need proof), $f'(\phi) > 0$ is the correct solution for most of the case.

Only when the θ_{cm} is too small, so that the e-z line bended so much. In the following plot, the $E_x \in (0, 15)$ MeV, $\theta_{cm} \in (0°, 60°)$. There is region where double solution exists, in that case, at the boundary that manifold is folded, the proper solution takes $f'(\phi) < 0, \phi\, Z > 0$.



It seems (Need proof) that the fold happens when

$$\gamma^2\beta^2k^2 - (\gamma q - y)^2 = 0$$

To numerically find the solution, a newton's method is adequacy.

$$\phi_{i+1} = \phi_i - \frac{f(\phi_i)}{f'(\phi_i)}$$

Appendix

Lorentz Transform with boost $\vec{\beta}$

$$\mathbb{P} = \begin{pmatrix} E \\ \vec{k} \end{pmatrix} \to \mathbb{P}' = \begin{pmatrix} \gamma E + \gamma \vec{\beta} \cdot \vec{k} \\ \gamma E \vec{\beta} + \vec{k} + (\gamma - 1)(\hat{\beta} \cdot \vec{k})\hat{\beta} \end{pmatrix} = \begin{pmatrix} \gamma E + \gamma \beta k \cos\theta \\ (\gamma \beta E + \gamma k \cos\theta)\hat{\beta} + k \sin\theta\, \hat{n} \end{pmatrix}$$

where $\hat{\beta} \perp \hat{n}$.

Kinematics of 2-body scattering

Suppose the reaction is labeled as b(a,1,2), where a→1, b→2 after scattering. The four momenta of the incident channel are

$$\mathbb{P}_a = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} \\ \overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}_b = \begin{pmatrix} m_b \\ \vec{0} \end{pmatrix}$$

The center of mass 4-vector is

$$\mathbb{P}_c = \mathbb{P}_a + \mathbb{P}_b = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} + m_b \\ \overrightarrow{k_a} \end{pmatrix} = \begin{pmatrix} E_c \\ \overrightarrow{k_a} \end{pmatrix}$$

The system mass is $M_c = \sqrt{E_c^2 - k_a^2} = \sqrt{m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2}}$

The Lorentz boost vector is

$$\vec{\beta} = \frac{\overrightarrow{k_a}}{E_c}, \qquad \gamma = \frac{E_c}{M_c}, \qquad \gamma\vec{\beta} = \frac{\overrightarrow{k_a}}{M_c}$$

The system undergoes a Lorentz boot, so that the total momentum from $\vec{0}$ to $\overrightarrow{k_a} = \gamma\vec{\beta}M_c$, to see that, in the CM frame,

$$\mathbb{P}'_c = \begin{pmatrix} \gamma E_c - \gamma\vec{\beta} \cdot \overrightarrow{k_a} \\ -\gamma\vec{\beta}E_c + \gamma\overrightarrow{k_a} \end{pmatrix} = \begin{pmatrix} M_c \\ \vec{0} \end{pmatrix}$$

where

$$\gamma E_c - \gamma\vec{\beta} \cdot \overrightarrow{k_a} = \gamma E_c - \gamma\frac{k_a^2}{E_c} = \frac{\gamma}{E_c}(M_c^2) = M_c$$

In CM frame,

$$\mathbb{P}'_a = \begin{pmatrix} \gamma\sqrt{m_a^2+k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ -\gamma\vec{\beta}\sqrt{m_a^2+k_a^2} + \gamma\overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}'_b = \begin{pmatrix} \gamma m_b \\ -\gamma\vec{\beta} m_b \end{pmatrix}$$

Check the energy part

$$\gamma\sqrt{m_a^2+k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} + \gamma m_b = \gamma E_c - \gamma\vec{\beta}\cdot\overrightarrow{k_a} = M_c$$

The momentum part

$$-\gamma\vec{\beta}\sqrt{m_a^2+k_a^2} + \gamma\overrightarrow{k_a} - \gamma\vec{\beta} m_b = \gamma\vec{\beta}E_c + \gamma\overrightarrow{k_a} = 0$$

After the scattering, only direction changed.

$$p^2 = \frac{1}{4M_c^2}\left(M_c^2 - (m_1+m_2)^2\right)\left(M_c^2 - (m_1-m_2)^2\right)$$

The 4-momenta

$$\mathbb{P}'_1 = \begin{pmatrix} \sqrt{m_1^2+p^2} \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}'_2 = \begin{pmatrix} \sqrt{m_2^2+p^2} \\ -\vec{p} \end{pmatrix}$$

Return to the Lab frame

$$\mathbb{P}'_1 = \begin{pmatrix} \gamma\sqrt{m_1^2+p^2} + \gamma\vec{\beta}\cdot\vec{p} \\ \gamma\sqrt{m_1^2+p^2}\vec{\beta} + \vec{p} + (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}, \qquad \mathbb{P}'_2 = \begin{pmatrix} \gamma\sqrt{m_2^2+p^2} - \gamma\vec{\beta}\cdot\vec{p} \\ \gamma\sqrt{m_2^2+p^2}\vec{\beta} - \vec{p} - (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}$$

The momentum part can be rewritten using $\hat{\beta}\cdot\vec{p} = p\ \cos\theta$

$$\mathbb{P}'_1 = \begin{pmatrix} \gamma\sqrt{m_1^2+p^2} + \gamma\beta p\cos\theta \\ \left(\gamma\beta\sqrt{m_1^2+p^2} + \gamma p\cos\theta\right)\hat{\beta} + p\sin\theta\ \hat{n} \end{pmatrix}, \mathbb{P}'_2 = \begin{pmatrix} \gamma\sqrt{m_2^2+p^2} - \gamma\beta p\cos\theta \\ \left(\gamma\beta\sqrt{m_2^2+p^2} - \gamma p\cos\theta\right)\hat{\beta} - p\sin\theta\ \hat{n} \end{pmatrix}$$

where $\hat{\beta}\perp\hat{n}$.

The total energy must be conserved,

$$E_1 + E_2 = \gamma\sqrt{m_1^2+p^2} + \gamma\sqrt{m_2^2+p^2} = \sqrt{m_a^2+k_a^2} + m_b$$

The opening angle

$$k_1 k_2 \cos\theta_{12} = \left(\gamma\beta\sqrt{m_1^2 + p^2} + \gamma p \cos\theta\right)\left(\gamma\beta\sqrt{m_2^2 + p^2} - \gamma p \cos\theta\right) - p^2 \sin^2\theta$$

Assume the masses of 1,2 are equal the masses of a, b

After the scattering, only direction changed. Also, we can check the momentum formula

$$p^2 = \frac{1}{4M_c^2}(M_c^2 - (m_a + m_b)^2)(M_c^2 - (m_a - m_b)^2) \overset{?}{\rightarrow} \gamma^2\beta^2 m_b^2$$

Each term,

$$M_c^2 - (m_a + m_b)^2 = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - (m_a + m_b)^2 = 2m_b\sqrt{m_a^2 + k_a^2} - 2m_a m_b$$

$$M_c^2 - (m_a - m_b)^2 = m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - (m_a - m_b)^2 = 2m_b\sqrt{m_a^2 + k_a^2} + 2m_a m_b$$

$$p^2 = \frac{1}{4M_c^2}\left(4m_b^2(m_a^2 + k_a^2) - 4m_a^2 m_b^2\right) = m_b^2 \frac{k_a^2}{M_c^2} = \gamma^2\beta^2 m_b^2$$

Assume the mass of a and b are the same

Suppose the masses $m_a = m_b = m = m_1 = m_2$

$$\mathbb{P}'_a = \begin{pmatrix} \gamma\sqrt{m^2 + k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ -\gamma\vec{\beta}\sqrt{m^2 + k_a^2} + \gamma\overrightarrow{k_a} \end{pmatrix}, \qquad \mathbb{P}'_b = \begin{pmatrix} \gamma m \\ -\gamma\vec{\beta} m \end{pmatrix}$$

$$p^2 = \frac{M_c^2}{4} - m^2 = \gamma^2\beta^2 m^2 \Longrightarrow M_c^2 = 4(\gamma^2\beta^2 + 1)m^2 = 4\gamma^2 m^2$$

$$M_c = 2\gamma m$$

As we expected, as the particle a and b should share equal energy in CM frame, i.e.

$$\gamma\sqrt{m^2 + k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} = \gamma m$$

Which can be obtained using

$$m^2 = \left(\gamma\sqrt{m^2 + k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a}\right)^2 - \gamma^2\beta^2 m^2$$

The scattered 4-momenta in CM frame are

$$\mathbb{P}'_1 = \begin{pmatrix} \gamma m \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}'_2 = \begin{pmatrix} \gamma m \\ -\vec{p} \end{pmatrix}$$

In Lab frame,

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m + \gamma\vec{\beta}\cdot\vec{p} \\ \gamma^2\vec{\beta}m + \vec{p} + (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \gamma^2 m - \gamma\vec{\beta}\cdot\vec{p} \\ \gamma^2\vec{\beta}m - \vec{p} - (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} \end{pmatrix}$$

The momentum part can be rewritten using $\hat{\beta}\cdot\vec{p} = p\ \cos\theta = \gamma\beta m\cos\theta$

$$\gamma^2\vec{\beta}m + \vec{p} + (\gamma-1)(\hat{\beta}\cdot\vec{p})\hat{\beta} = (\gamma^2\beta m + \gamma p\cos\theta)\hat{\beta} + p\sin\theta\ \hat{n}$$

Where $\hat{\beta}\perp\hat{n}$.

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m(1+\beta^2\cos\theta) \\ \gamma^2\beta m(1+\cos\theta)\hat{\beta} + p\sin\theta\ \hat{n} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \gamma^2 m(1-\beta^2\cos\theta) \\ \gamma^2\beta m(1-\cos\theta)\hat{\beta} - p\sin\theta\ \hat{n} \end{pmatrix}$$

When scattering angle $\theta = 0$

$$\mathbb{P}_1 = \begin{pmatrix} \gamma^2 m(1+\beta^2) \\ 2\gamma^2 m\,\vec{\beta} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$$

Check:

$$\gamma^2 m(1+\beta^2) = \gamma^2 m + (\gamma^2-1)m = 2\gamma^2 m - m = \gamma M_c - m = E_c - m = \sqrt{m^2 + k_a^2}$$

$$2\gamma^2 m\,\vec{\beta} = 2\gamma m\frac{\overrightarrow{k_a}}{M_c} = \overrightarrow{k_a}$$

The total energy or energy conservation,

$$E_1 + E_2 = \gamma^2 m(1+\beta^2\cos\theta) + \gamma^2 m(1-\beta^2\cos\theta) = 2\gamma^2 m = E_c$$

The opening angle

$$k_1 k_2\cos\theta_{12} = \gamma^4\beta^4 m^2\sin^2\theta$$

Assume the mass of 1 and 2 becomes equal after scattering

Suppose the mass becomes m, the momentum is

$$p^2 = \frac{M_c^2}{4} - m^2 = \frac{1}{4}\left(m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - 4m^2\right)$$

$$\mathbb{P}_1' = \begin{pmatrix} \frac{M_c}{2} \\ \vec{p} \end{pmatrix}, \qquad \mathbb{P}_2' = \begin{pmatrix} \frac{M_c}{2} \\ -\vec{p} \end{pmatrix}$$

$$\mathbb{P}_1 = \begin{pmatrix} \frac{\gamma M_c}{2} + \gamma\beta p\cos\theta \\ \left(\frac{\gamma\beta M_c}{2} + \gamma p\cos\theta\right)\hat{\beta} + p\sin\theta\,\hat{n} \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \frac{\gamma M_c}{2} - \gamma\beta p\cos\theta \\ \left(\frac{\gamma\beta M_c}{2} - \gamma p\cos\theta\right)\hat{\beta} - p\sin\theta\,\hat{n} \end{pmatrix}$$

Opening angle

$$k_1 k_2 \cos\theta_{12} = \frac{\gamma^2\beta^2 M_c^2}{4} - \gamma^2\beta^2 p^2\cos^2\theta - p^2$$

$$= \frac{\gamma^2\beta^2 M_c^2}{4} - \gamma^2\beta^2 p^2\cos^2\theta - p^2 = \frac{\gamma^2\beta^2 M_c^2}{4} - \frac{M_c^2}{4} + m^2 - \gamma^2\beta^2 p^2\cos^2\theta$$

Kinematics of transfer reaction

Kinematics of knockout reaction